

AMCS 608

Problem set 10 due December 13, 2010

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Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari. Conformal Mapping is an especially good reference for material on harmonic functions.

Standard problem: The following problems should be done, but do not have to be handed in.

1. This was done in class, but you should review it to be sure you understand the basic properties of Möbius transformations. Let a, b, c, d be complex numbers with $ad - bc \neq 0$, and define

$$f(z) = \frac{az + b}{cz + d}. \quad (1)$$

Maps of this type are called Möbius transformations. There are three basic Möbius transformations:

- Translations $z \mapsto z + a$.
- Dilations $z \mapsto az$.
- Inversion $z \mapsto z^{-1}$.

Prove that every Möbius transformation is a composition of maps of these three types. Let $\gamma \subset \mathbb{C}$ be either a circle or a straight-line, and let f be a Möbius transformation. Show that $f(\gamma)$ is either a circle or a straight line.

2. Suppose that u is a real valued harmonic function defined in the annulus $A = \{z : \rho < |z| < R\}$. Show that there exist bi-infinite sequences of real numbers $\{a_n, b_n\}$ so that if $z = re^{i\theta}$, then we have a locally uniformly convergent expansion:

$$u(z, \bar{z}) = a_0 + b_0 \log r + \sum_{n=1}^{\infty} [(a_n r^n + a_{-n} r^{-n}) \cos(n\theta) + (b_n r^n + b_{-n} r^{-n}) \sin(n\theta)] \quad (2)$$

Here $z = re^{i\theta}$. Give a condition, in terms of this expansion, for u to have a single valued harmonic conjugate in A .

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. The map $j(z) = z + \frac{1}{z}$ is very useful for studying flows. For $\rho > 1$, show that the image under j of the circle $\{z : |z| = \rho\}$ is an ellipse with semi-axes $a = \rho + \rho^{-1}$, and $b = \rho - \rho^{-1}$.
 - (a) What is the image of $D_\rho(0)^c$?
 - (b) Is the j a one-to-one map on this set?
 - (c) What happens when $\rho = 1$?

2. Recall that a continuous function, u defined in a neighborhood of $\overline{D}_r(z_0)$, satisfies the mean value property if, for $\rho < r$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}, \bar{z}_0 + e^{-i\theta}) d\theta. \quad (3)$$

Suppose that u is a continuous function defined in a connected domain D , which satisfies the mean value property. Show that u satisfies the maximum principle: if u attains a local interior maximum, then u is constant. Use this observation to show that u is actually harmonic. Hint: Being harmonic is a local property, so it is enough to consider u on disks with $\overline{D}_r(z_0) \subset D$.

3. Suppose that D is a simply connected domain in \mathbb{C} for which there is an 1-1, onto analytic map $f : D \rightarrow D_1(0)$, which extends to be a 1-1, onto, C^1 -map from \overline{D} to $\overline{D}_1(0)$. Let $g \in C^0(bD)$; in class we showed that

$$u(z, \bar{z}) = \frac{1}{2\pi} \int_{bD} g(w) \frac{1 - |f(z)|^2}{|f(w) - f(z)|^2} \cdot \frac{f'(w)dw}{if(w)}, \quad (4)$$

solves Dirichlet's problem in D ,

$$\begin{cases} \Delta u = 0 \text{ in } D, \\ u|_{bD} = g. \end{cases} \quad (5)$$

Give a geometric interpretation for the measure

$$ds = \frac{f'(w)dw}{if(w)} \text{ for } w \in bD, \quad (6)$$

which explains why it is real. If γ is a connected arc of bD , then describe, in simple geometric terms, the value of the integral:

$$\frac{1}{2\pi} \int_{\gamma} \frac{f'(w)dw}{if(w)}. \quad (7)$$

4. In the following problems, even though one of the regions is not bounded, the formula of the previous problem can still be applied, explain why.

(a) Show that the map $z \mapsto z^2$ carries the positive quadrant onto the upper half plane. Find a formula for the bounded solution of the Dirichlet problem in the positive quadrant, with bounded continuous data specified along the union of the positive x - and y -axes. Are there harmonic functions in this region, which vanish on the boundary of the positive quadrant?

(b) Show that the map $z \mapsto e^z$ carries the strip $S = \{z : 0 < \text{Im } z < \pi\}$ 1-1, onto the upper half plane. Find a formula for the solution to Dirichlet problem in S , where the data is a bounded function $g \in C^0(bS)$. Note that bS has two connected components. What if $\lim_{x \rightarrow -\infty} g(x) = M_-$, $\lim_{x \rightarrow -\infty} g(x+i\pi) = M_+$ both exist. For $0 < y < \pi$, what is

$$\lim_{x \rightarrow -\infty} u(x + iy)? \quad (8)$$

Hint: What is

$$\lim_{r \rightarrow 0^+} \int_0^{\infty} \frac{r \sin \theta da}{(r \cos \theta - a)^2 + r^2 \sin^2 \theta}. \quad (9)$$

5. The Dirichlet problem for the Laplace equation can be solved by conformal mapping.

(a) For the domain

$$E_r = \{z : r < |z| < 1\}, \quad (10)$$

find a harmonic function u in E_r so that

$$u(re^{i\theta}) = 1 \text{ and } u(e^{i\theta}) = 0. \quad (11)$$

(b) For $a \in (0, \frac{1}{2})$ and $a < b < 1 - a$, let

$$D = \{z : |z| < 1 \text{ and } |z - a| > b\}. \quad (12)$$

Find a harmonic function v in D so that

$$v(a + be^{i\theta}) = 1 \text{ and } v(e^{i\theta}) = 0. \quad (13)$$

(c) Compute

$$\int_{E_r} |\nabla u|^2 dx dy \text{ and } \int_D |\nabla v|^2 dx dy. \quad (14)$$

6. Suppose that D_1 and D_2 are the complements of bounded regions, which are simply connected in $\mathbb{C} \cup \{\infty\}$. For any ϕ show that there is a *unique* conformal map $f : D_1 \rightarrow D_2$, which satisfies

$$f(z) \rightarrow \infty \text{ and } f'(z) \rightarrow re^{i\phi} \text{ as } z \rightarrow \infty. \quad (15)$$

Here $r \in (0, \infty)$. Prove that f is uniquely determined by these conditions. Let

$$D_\theta = \mathbb{C} \setminus \{te^{i\theta} : t \in [-2, 2]\}. \quad (16)$$

Find a conformal map $f : D_\theta \rightarrow D_0$, satisfying (15) with $\phi = 0$.

7. Suppose that $f : D_1 \rightarrow D_2$ is a conformal map, and that $W = (X(u, v), Y(u, v))$ is a vector field defined in D_2 , which satisfies

$$\nabla \cdot W = \nabla \times W = 0. \quad (17)$$

We can identify W with a complex valued function $X + iY$. Show that the vector field in D_1 defined by the complex valued function

$$\tilde{X}(z) + i\tilde{Y}(z) = (X(f(z)) + iY(f(z)))\overline{f'(z)}$$

satisfies (17) in D_1 . Show that if (X, Y) is tangent to the boundary of D_2 , then (\tilde{X}, \tilde{Y}) is tangent to the boundary of D_1 .

8. The map $j(z) = z + \frac{1}{z}$ is very useful for studying flows. For $\rho > 1$, show that the image under j of the circle $\{z : |z| = \rho\}$ is an ellipse with semi-axes $a = \rho + \rho^{-1}$, and $b = \rho - \rho^{-1}$. Use this to find the flow past the ellipse

$$E_{a,b} = \{(x, y) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\}, \quad (18)$$

with velocity $(1, 0)$ at infinity, and zero circulation. Find the flows with non-zero circulation as well.

9. Suppose that f is a non-constant, meromorphic function from \mathbb{C} to \mathbb{C} that satisfies

$$f(z + 1) = f(z) \text{ and } f(z + i) = f(z). \quad (19)$$

- (a) Prove that for any $m, n \in \mathbb{Z}$ we also have $f(z + m + ni) = f(z)$. Such a function is said to be doubly periodic.
- (b) Show that f must have poles, that is a doubly periodic, analytic function is constant.
- (c) Let R be the square with vertices $0, 1, i, 1 + i$. We can suppose that the poles and zeros of f do not lie on ∂R . Show that

$$\int_{\partial R} f(z) dz = 0 \text{ and } \int_{\partial R} \frac{f'(z)}{f(z)} dz = 0 \quad (20)$$

- (d) Show that the equation $f(z) = 0$ has at least two solutions, counted with multiplicity, in R .