

AMCS 608

Problem set 2 due September 28, 2010

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Reading: References for this material are *Principles of Mathematical Analysis*, by Walter Rudin, *The Way of Analysis* by Robert Strichartz, *Elementary Classical Analysis*, by Jerrold Marsden, *Calculus on Manifolds* by Michael Spivak, and *Linear Algebra*, by Peter Lax.

Standard problems: The following problems should be done, but do not have to be handed in. You should get together and do these problems with your classmates; it will help you to get the basic notions of topology clear in your heads.

1. Suppose that $\langle x_n \rangle$ is a sequence of real numbers with

$$\lim_{n \rightarrow \infty} x_n = L. \quad (1)$$

If we define $X_n = \frac{x_1 + \dots + x_n}{n}$, then $\lim_{n \rightarrow \infty} X_n = L$ as well. Find an example of non-convergent sequence $\langle x_n \rangle$ such that $\langle X_n \rangle$ is convergent.

2. Show that the two definitions of closed set are equivalent: A subset C of a metric space (X, d) contains all of its limit points if and only if its complement C^c is an open set.
3. Let (X, d) be a metric space and $f : X \rightarrow \mathbb{R}$ a continuous function. Prove that

$$f^{-1}(0) = \{x \in X : f(x) = 0\} \quad (2)$$

is a closed set.

4. Show that the only path connected subsets of \mathbb{R} are rays and intervals.
5. A subset S of \mathbb{R}^n is connected if it cannot be written as a union:

$$S = (U \cap S) \cup (V \cap S) \quad (3)$$

where U and V are disjoint, open subsets of \mathbb{R}^n , with non-empty intersection with S . Show that an open subset of \mathbb{R}^n is path connected if and only if it is connected.

6. Prove that the set $\{x : \|x - a\| < r\}$ is an open subset of \mathbb{R}^n by considering the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $f(x) = \|x - a\|$.
7. (a) If $A \subset \mathbb{R}^n$ is closed and $x \notin A$, then prove that there is a number $d > 0$ so that $\|y - x\| \geq d$, for all $y \in A$.
- (b) If A is closed, B is compact, and $A \cap B = \emptyset$, then prove that there is a $d > 0$ so that $\|x - y\| \geq d$, for all $x \in A$ and $y \in B$.
- (c) Find two closed, non-compact subsets $A, B \subset \mathbb{R}^2$ so that the previous statement is false.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $U \subset \mathbb{R}^n$ be connected, and $f : U \rightarrow \mathbb{N}$, a continuous function. Show that f is constant.
2. Let $S \subset (X, d)$ a metric space, and let S' denote the limits of sequences contained in S . That is, $s \in S'$ if there is a sequence $\langle s_n \rangle \subset S$, such that

$$\lim_{n \rightarrow \infty} s_n = s. \quad (4)$$

Show that $\bar{S} = S \cup S'$ is a closed set. This set is called the *closure* of S . Show that

$$\bar{S} = \bigcap_{\{S \subset C : C \text{ is closed.}\}} C. \quad (5)$$

3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a uniformly continuous function. Show that there are constants C_1, C_2 so that

$$|f(x)| \leq C_1 + C_2 \|x\|. \quad (6)$$

Produce an example to show that the converse is false.

4. To define the metric on the circle S^1 we identify it with the subset of \mathbb{R}^2 :

$$S_1^1 = \{(x, y) : x^2 + y^2 = 1\}, \quad (7)$$

and use the metric induced from the embedding:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \quad (8)$$

Let $f : S^1 \rightarrow \mathbb{R}$ be a continuous function. Show that there is a point $(x, y) \in S_1^1$ so that

$$f(x, y) = f(-x, -y). \quad (9)$$

5. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. Show that f has a fixed point, that is, an x so that $f(x) = x$. Is this still true if $[0, 1]$ is replaced by $(0, 1]$?
6. Suppose that there is a constant C so that $f : [0, 1] \rightarrow \mathbb{R}$ satisfies the estimate:

$$|f(x) - f(y)| \leq C|x - y|^2. \quad (10)$$

Show that f is constant.

7. Suppose that $\{C_0, C_1, \dots, C_{n-1}, C_n\}$ are real constants such that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0. \quad (11)$$

Show that the equation $C_0 + C_1x + \dots + C_{n-1}x^{n-1} + C_nx^n = 0$ has a solution in $(0, 1)$. Is this still true if the coefficients $\{C_j\}$ are allowed to take complex values?