

AMCS 608

Problem set 3 due October 5, 2010

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Reading: References for this material are *Principles of Mathematical Analysis*, by Walter Rudin, *The Way of Analysis* by Robert Strichartz, *Elementary Classical Analysis*, by Jerrold Marsden, *Calculus on Manifolds*, by Michael Spivak, and *Linear Algebra*, by Peter Lax.

Standard problems: The following problems should be done, but do not have to be handed in.

1. Suppose that $f(x)$ is a Riemann integrable function defined on $[a, b]$. Prove that $|f(x)|$ is also Riemann integrable, and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx. \quad (1)$$

2. Define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{(1-x)^2}} e^{-\frac{1}{(1+x)^2}} & \text{for } -1 < x < 1 \\ 0 & \text{for } |x| \geq 1. \end{cases} \quad (2)$$

Prove that f is a non-negative, infinitely differentiable function on \mathbb{R} , with support equal to $[-1, 1]$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let f be a bounded function defined on $[a, b]$. Show that f is Riemann integrable if and only if, for every $\epsilon > 0$, there is a partition P so that

$$U(f, P) - L(f, P) < \epsilon. \quad (3)$$

2. Define the functions

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases} \quad g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases} \quad (4)$$

Prove that f is not Riemann integrable over $[0, 1]$, but that g is. What is

$$\int_0^1 g(x) dx? \quad (5)$$

3. Suppose that f and g are Riemann integrable functions on $[0, 1]$. Prove that $f \cdot g$ is also Riemann integrable on $[0, 1]$. From the Cauchy-Schwarz inequality for finite sums to deduce that

$$\left| \int_0^1 f(x)g(x) dx \right| \leq \sqrt{\int_0^1 |f(x)|^2 dx} \sqrt{\int_0^1 |g(x)|^2 dx}. \quad (6)$$

4. Suppose that f is a continuous function on $[0, 1]$, and define

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x} dx. \quad (7)$$

Use the uniform continuity of f and the fact that, for any $a \in \mathbb{R}$,

$$\int_a^{a+\frac{1}{n}} e^{-2\pi i n x} dx = 0, \quad (8)$$

to prove that $\lim_{|n| \rightarrow \infty} \hat{f}(n) = 0$.

5. Suppose that w is a non-negative function, Riemann integrable on $[a, b]$ and $f \in \mathcal{C}^0([a, b])$. Show that there is a $c \in (a, b)$ so that

$$\int_a^b f(x)w(x) dx = f(c) \int_a^b w(x) dx. \quad (9)$$

6. Suppose that f is a continuously differentiable function on $[0, 1]$ for which $f(0) = f'(0) = f(1) = f'(1) = 0$. Show that the extension of f to all of \mathbb{R} defined by letting $f(x) = 0$ for $x \notin [0, 1]$ is a continuously differentiable function. Let $h_n(x) = c_n(1 - x^2)^n \chi_{[-1,1]}(x)$, with c_n chosen so that

$$\int_{-1}^1 h_n(x) dx = 1. \quad (10)$$

Show that for $x \in [0, 1]$

$$p_n(x) = \int_{-\infty}^{\infty} h_n(x - y) f(y) dy, \quad (11)$$

defines a sequence of polynomials such that $\langle p_n \rangle$ converges uniformly to f and $\langle p'_n \rangle$ converges uniformly to f' . Hint: Observe that:

$$p_n(x) = \int_{-\infty}^{\infty} h_n(y) f(x - y) dy. \quad (12)$$