

AMCS 608

Problem set 3 due October 6, 2009

Dr. Epstein

Reading: References for this material are *Principles of Mathematical Analysis*, by Walter Rudin, *The Way of Analysis* by Robert Strichartz, *Elementary Classical Analysis*, by Jerrold Marsden, *Calculus on Manifolds*, by Michael Spivak, and *Linear Algebra*, by Peter Lax.

Standard problems: The following problems should be done, but do not have to be handed in.

1. Suppose that $f(x)$ is a Riemann integrable function defined on $[a, b]$. Prove that $|f(x)|$ is also Riemann integrable, and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx. \quad (1)$$

2. Suppose that $f(x, y)$, defined in $[0, 1] \times [0, 1]$, has partial derivatives in $(0, 1) \times (0, 1)$. If $\partial_y f(x, y) = 0$ throughout $(0, 1) \times (0, 1)$, show that there is a differentiable function $g(x)$ so that $f(x, y) = g(x)$. How is $\partial_x f(x, y)$ related to $g'(x)$?
3. Define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{(1-x)^2}} e^{-\frac{1}{(1+x)^2}} & \text{for } -1 < x < 1 \\ 0 & \text{for } |x| \geq 1. \end{cases} \quad (2)$$

Prove that f is a non-negative, infinitely differentiable function on \mathbb{R} , with support equal to $[-1, 1]$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that f and g are Riemann integrable functions on $[0, 1]$. Prove that $f \cdot g$ is also Riemann integrable on $[0, 1]$. From the Cauchy-Schwarz inequality for finite sums to deduce that

$$\left| \int_0^1 f(x)g(x) dx \right| \leq \sqrt{\int_0^1 |f(x)|^2 dx} \sqrt{\int_0^1 |g(x)|^2 dx}. \quad (3)$$

2. Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ denote a linear transformation. If $\|\cdot\|_1, \|\cdot\|_2$ are norms on \mathbb{R}^n and \mathbb{R}^m , respectively, then we define the norm of A to be

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_1}. \quad (4)$$

- (a) Show that $\|A\|$ is finite and, if $x \in \mathbb{R}^n$, then $\|Ax\|_2 \leq \|A\| \|x\|_1$.
 (b) If $m = n$, then show that A is invertible if and only if there is a positive constant C so that

$$\|Ax\|_1 \geq C \|x\|_1. \quad (5)$$

- (c) Suppose that $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear and invertible. Show that there is an $\epsilon > 0$, so that if $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear and $\|B\| < \epsilon$, then $A + B$ is invertible.
 (d) Let $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy $\|B\| < 1$; then show that, for any $v \in \mathbb{R}^n$, the infinite sum

$$Cv = \sum_{j=0}^{\infty} B^j v \quad (6)$$

converges. Show that $v \rightarrow Cv$ is a linear map, which satisfies $C = (\text{Id} - B)^{-1}$, and that

$$\|C\| \leq \frac{1}{1 - \|B\|}. \quad (7)$$

3. Let $\langle x, y \rangle$ be an inner product on \mathbb{R}^n . We define a function $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, by $F(x, y) = \langle x, y \rangle$. Show that F is differentiable and compute $DF(x, y)(a, b)$.
 4. Suppose that f is a continuously differentiable function on $[0, 1]$ for which $f(0) = f'(0) = f(1) = f'(1) = 0$. Show that the extension of f to all of \mathbb{R} defined by letting $f(x) = 0$ for $x \notin [0, 1]$ is a continuously differentiable function. Let $h_n(x) = c_n(1 - x^2)^n \chi_{[-1, 1]}(x)$, with c_n chosen so that

$$\int_{-1}^1 h_n(x) dx = 1. \quad (8)$$

Show that for $x \in [0, 1]$

$$p_n(x) = \int_{-\infty}^{\infty} h_n(x - y) f(y) dy, \quad (9)$$

is a sequence of polynomials such that $\langle p_n \rangle$ converges uniformly to f and $\langle p'_n \rangle$ converges uniformly to f' . Hint: Observe that:

$$p_n(x) = \int_{-\infty}^{\infty} h_n(y) f(x-y) dy. \quad (10)$$

5. Let g be a continuous function on the unit circle in \mathbb{R}^2 that satisfies: $g(1, 0) = g(0, 1) = 0$ and $g(-x) = -g(x)$. Define

$$f(x) = \begin{cases} \|x\| g\left(\frac{x}{\|x\|}\right), & \text{for } x \neq (0, 0) \\ 0, & \text{for } x = (0, 0) \end{cases} \quad (11)$$

For $x \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and $t \in \mathbb{R}$, define $h(t) = f(tx)$.

- Show that h is differentiable.
 - Prove that f is not differentiable at $(0, 0)$ unless $g \equiv 0$.
 - Show that the directional derivatives $D_x f(0, 0)$ exist for all directions x , but that $D_{x+y} f(0, 0) = D_x f(0, 0) + D_y f(0, 0)$ is not true for all x and y .
6. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \quad (12)$$

- Show that $\partial_x f(0, y) = -y$ and $\partial_y f(x, 0) = x$.
 - Show that $\partial_x \partial_y f(0, 0)$ and $\partial_y \partial_x f(0, 0)$ exist but are not equal.
 - Explain why this does not contradict the theorem proved in class.
7. As noted in class, we can think of the j th derivative of a function f as a j -linear function, $D^j f(x)(v_1, \dots, v_j)$. Suppose that f is a j -times differentiable, real valued function in $B_r(a)$. Show that

$$\frac{d^j f}{dt^j}(x + tv)|_{t=0} = D^j f(x)(v, \dots, v). \quad (13)$$

Use this to show that if f is three times differentiable, then

$$f(x) = f(y) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)(x_j - y_j) + \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k}(x)(x_j - y_j)(x_k - y_k) + \mathcal{O}(\|x - y\|^3). \quad (14)$$