

AMCS 608

Problem set 4 due October 14, 2010

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Reading: References for this material are *Principles of Mathematical Analysis*, by Walter Rudin, *The Way of Analysis* by Robert Strichartz, *Elementary Classical Analysis*, by Jerrold Marsden, *Calculus on Manifolds*, by Michael Spivak, and *Linear Algebra*, by Peter Lax.

Standard Problem Solution does not have to be handed in:

1. Suppose that $f(x, y)$, defined in $[0, 1] \times [0, 1]$, has partial derivatives in $(0, 1) \times (0, 1)$. If $\partial_y f(x, y) = 0$ throughout $(0, 1) \times (0, 1)$, show that there is a differentiable function $g(x)$ so that $f(x, y) = g(x)$. How is $\partial_x f(x, y)$ related to $g'(x)$?

2. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \quad (1)$$

- (a) Show that $\partial_x f(0, y) = -y$ and $\partial_y f(x, 0) = x$.
 - (b) Show that $\partial_x \partial_y f(0, 0)$ and $\partial_y \partial_x f(0, 0)$ exist but are not equal.
 - (c) Explain why this does not contradict the theorem proved in class.
3. As noted in class, we can think of the j th derivative of a function f as a j -linear function, $D^j f(x)(v_1, \dots, v_j)$. Suppose that f is a j -times differentiable, real valued function in $B_r(a)$. Show that

$$\frac{d^j f}{dt^j}(x + tv) \Big|_{t=0} = D^j f(x)(v, \dots, v). \quad (2)$$

Use this to show that if f is three times differentiable, then

$$f(x) = f(y) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)(x_j - y_j) + \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k}(x)(x_j - y_j)(x_k - y_k) + O(\|x - y\|^3). \quad (3)$$

4. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable. Show that f is not one-to-one. Hint: If, for example, $\partial_x f(x, y)$ is not zero, then show that the map $F : (x, y) \rightarrow (f(x, y), y)$ is locally one-to-one and onto.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

- Let $\langle x, y \rangle$ be an inner product on \mathbb{R}^n . We define a function $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, by $F(x, y) = \langle x, y \rangle$. Show that F is differentiable and compute $DF(x, y)(a, b)$.
- Suppose that $U \subset \mathbb{R}^n$ is an open convex set, and $f : U \rightarrow \mathbb{R}^m$ is a continuously differentiable function. Show that for $x_1, x_2 \in U$,

$$f(x_1) - f(x_2) = \int_0^1 Df(tx_1 + (1-t)x_2)(x_1 - x_2)dt. \quad (4)$$

Use this to show that in any compact subset $K \subset U$, and for any $\epsilon > 0$, there is a $\delta > 0$ so that if $x_1, x_2 \in K$ and $\|x_1 - x_2\| < \delta$, then

$$\|f(x_1) - f(x_2) - Df(x_1)(x_1 - x_2)\| \leq \epsilon \|x_1 - x_2\|. \quad (5)$$

- Let g be a continuously differentiable function on the unit circle in \mathbb{R}^2 , which satisfies: $g(1, 0) = g(0, 1) = 0$ and $g(-x) = -g(x)$. Define

$$f(x) = \begin{cases} \|x\| g\left(\frac{x}{\|x\|}\right), & \text{for } x \neq (0, 0) \\ 0, & \text{for } x = (0, 0) \end{cases} \quad (6)$$

- Show that for each fixed $x \in \mathbb{R}^2$, the function $t \mapsto f(tx)$ is differentiable.
 - Prove that $f(x)$ is not differentiable at $(0, 0)$ unless $g \equiv 0$.
- Let $U \subset \mathbb{R}^n$ be an open set and $f : U \rightarrow \mathbb{R}^n$ a continuously differentiable map, such that, at any every point $x \in U$, $\det Df(x) \neq 0$. Show that $f(U)$ is an open set.
 - Show that if $f(x, y)$ is a continuous function in $[0, 1] \times [0, 1]$, for which $\partial_x f(x, y)$ exists and is continuous in $(0, 1) \times [0, 1]$, then

$$g(x) = \int_0^1 f(x, y)dy \quad (7)$$

is a continuous function of $x \in [0, 1]$, which is a differentiable function of $x \in (0, 1)$. Moreover:

$$g'(x) = \int_0^1 \partial_x f(x, y) dy \quad (8)$$

6. Define a map from \mathbb{R}^2 to itself by setting

$$F(x, y) = (\sin x \cos y + \sin y \cos x, \cos x \cos y - \sin x \sin y). \quad (9)$$

Does there exist a point (x_0, y_0) such that F is locally invertible in a neighborhood of $F(x_0, y_0)$. You must prove your answer.

7. Suppose that F is a continuously differentiable function defined in a neighborhood of $(0, 0, 0)$ such that $F(0, 0, 0) = 0$, and each of the partial derivatives

$$F_x(0, 0, 0), F_y(0, 0, 0), F_z(0, 0, 0)$$

is non-zero. Show that there are differentiable functions, defined in a neighborhood of $(0, 0)$,

$$z = f(x, y), \quad x = g(y, z), \quad \text{and} \quad y = h(z, x),$$

solving $F(x, y, z) = 0$; that is

$$F(x, y, f(x, y)) = F(x, h(x, z), z) = F(g(y, z), y, z) = 0. \quad (10)$$

Show that

$$\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} \frac{\partial h}{\partial z} = -1, \quad (11)$$

at any point of $S = \{(x, y, z) : F(x, y, z) = 0\}$, sufficiently close to $(0, 0, 0)$. Explain this geometrically.