

AMCS 608

Problem set 4 due October 13, 2009

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Reading: References for this material are *Principles of Mathematical Analysis*, by Walter Rudin, *The Way of Analysis* by Robert Strichartz, *Elementary Classical Analysis*, by Jerrold Marsden, *Calculus on Manifolds*, by Michael Spivak, and *Linear Algebra*, by Peter Lax.

Standard Problem Solution does not have to be handed in: Suppose that $U \subset \mathbb{R}^n$ is an open convex set, and $f : U \rightarrow \mathbb{R}^m$ is a continuously differentiable function.

1. Show that for $x_1, x_2 \in U$,

$$f(x_1) - f(x_2) = \int_0^1 Df(tx_1 + (1-t)x_2)(x_1 - x_2)dt. \quad (1)$$

2. Use this to show that in any compact subset $K \subset U$, and for any $\epsilon > 0$, there is a $\delta > 0$ so that if $x_1, x_2 \in K$ and $\|x_1 - x_2\| < \delta$, then

$$\|f(x_1) - f(x_2) - Df(x_1)(x_1 - x_2)\| \leq \epsilon \|x_1 - x_2\|. \quad (2)$$

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable. Show that f is not one-to-one. Hint: If, for example, $\partial_x f(x, y)$ is not zero, then show that the map $F : (x, y) \rightarrow (f(x, y), y)$ is locally one-to-one and onto.
2. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \rightarrow \mathbb{R}^n$ a continuously differentiable map, such that, at any every point $x \in U$, $\det Df(x) \neq 0$. Show that $f(U)$ is an open set.
3. Show that if $f(x, y)$ is a continuous function in $[0, 1] \times [0, 1]$, for which $\partial_x f(x, y)$ exists and is continuous in $(0, 1) \times [0, 1]$, then

$$g(x) = \int_0^1 f(x, y)dy \quad (3)$$

is a continuous function of $x \in [0, 1]$, which is a differentiable function of $x \in (0, 1)$. Moreover:

$$g'(x) = \int_0^1 \partial_x f(x, y) dy \quad (4)$$

4. Let f be a continuously differentiable function defined in a neighborhood of $1 \in \mathbb{R}$, with $f(1) = 1$. What is a condition on f that assures that the equation

$$2f(xy) = f(x) + f(y) \quad (5)$$

can be solved for y as a function of x in a neighborhood of $(1, 1)$?

5. Define a map from \mathbb{R}^2 to itself by setting

$$F(x, y) = (\sin x \cos y + \sin y \cos x, \cos x \cos y - \sin x \sin y). \quad (6)$$

Does there exist a point (x_0, y_0) such that F is locally invertible in a neighborhood of $F(x_0, y_0)$. You must prove your answer.

6. Let $\{g_0(a), \dots, g_n(a)\}$ be continuously differentiable, real valued functions defined in a neighborhood of $0 \in \mathbb{R}$, and

$$p(a, x) = \sum_{j=0}^n g_j(a) x^j \quad (7)$$

- (a) Suppose that $g_0(0) = 0$, but $g_1(0) \neq 0$. Show that there is a $\delta > 0$, and an M , such that for $a \in (-\delta, \delta)$ the polynomial $x \rightarrow p(a, x)$ has a unique root $h(a)$ with $|h(a)| \leq M|a|$. Show that h is differentiable, and find a formula for $h'(0)$.
- (b) Suppose that A is a symmetric $n \times n$ matrix such that 0 is a simple eigenvalue of A . Suppose that B is an arbitrary $n \times n$ matrix. Show that for sufficiently small ϵ the matrix $A + \epsilon B$ has a simple eigenvalue, $\lambda(\epsilon)$ near to 0 . Can you prove that the eigenvector $v(\epsilon)$ is also a differentiable function of ϵ ? Hint: Consider $v(\epsilon) = v(0) + w(\epsilon)$, where $\langle v(0), w(\epsilon) \rangle = 0$.

7. Suppose that F is a continuously differentiable function defined in a neighborhood of $(0, 0, 0)$ such that $F(0, 0, 0) = 0$, and the implicit function theorem implies the existence of differentiable functions $z = f(x, y)$, $x = g(y, z)$ and $y = h(z, x)$, solving $F(x, y, z) = 0$ in a neighborhood of $(0, 0, 0)$. Show that

$$\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} \frac{\partial h}{\partial z} = -1, \quad (8)$$

at any point of $S = \{(x, y, z) : F(x, y, z) = 0\}$. Explain this geometrically.

8. Let $U \subset \mathbb{R}^n$ and let $f : U \rightarrow \mathbb{R}$ be a twice continuously differentiable function. Suppose that at $x_0 \in U$ we know that $\nabla f(x_0) = 0$ and the Hessian $\frac{\partial^2 f}{\partial x_j \partial x_k}(x_0)$ is invertible. Show that for a $\delta > 0$ there is a map $G : B_\delta(0) \rightarrow U$ so that $G(0) = x_0$ and $\nabla f(G(y)) = y$. Finally show that there is a function $g(y)$ defined in a neighborhood of 0 so that $G(y) = \nabla g(y)$.