

AMCS 608

Problem set 5 due October 26, 2010

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Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

Standard problems: The following problems should be done, but do not have to be handed in.

1. We can define polar coordinates in the complex plane by setting

$$x = r \cos \theta \quad y = r \sin \theta. \quad (1)$$

Show that in polar coordinates the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}. \quad (2)$$

2. Show that if f is analytic in $B_R(0)$ and $\operatorname{Re} f$, or $\operatorname{Im} f$ is a constant, then f is constant.
3. Suppose that g is an analytic function defined in $U \subset \mathbb{C}$ and f is an analytic defined in an open set containing $g(U)$. Show that $f \circ g$ is an analytic function, using just the definition of complex derivative. Show that the chain rule is valid:

$$(f \circ g)'(w) = f'(g(w))g'(w). \quad (3)$$

4. Using the definition of the complex contour integral evaluate the following integrals over γ , the semi-circle $\{z : |z| = 1, \operatorname{Im} z > 0\}$.

(a) $\int_{\gamma} \frac{dz}{z^n}$ where $n \in \mathbb{N}$.

(b) $\int_{\gamma} \bar{z}^n dz$, where $n \in \mathbb{Z}$.

(c) $\int_{\gamma} z^n \bar{z}^n dz$, where $n \in \mathbb{Z}$.

5. Let $f(x, y) = \sqrt{|x||y|}$, for all $(x, y) \in \mathbb{R}^2$. Show that f satisfies the Cauchy-Riemann equation at $(0, 0)$, but does not have a complex derivative at this point.
6. Prove the following:
 - (a) The power series $\sum_{n=1}^{\infty} nz^n$ converges for any z with $|z| < 1$, but does not converge for any point where $|z| = 1$.
 - (b) The power series $\sum_{n=1}^{\infty} z^n/n^2$ converges for any z with $|z| \leq 1$.
 - (c) The power series $\sum_{n=1}^{\infty} z^n/n$ converges for any z with $|z| \leq 1$, except $z = 1$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $\omega = f(x, y)dx + g(x, y)dy$ be a 1-form defined in $U \subset \mathbb{R}^2$ and $H(s, t) : V \rightarrow U$ a C^2 -map. Show that

$$H^*(d\omega) = d(H^*\omega). \quad (4)$$

Is it necessary for H to be 1-1?

2. Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be an imbedded C^1 -curve in the plane, and let $\mathbf{p} = \{p_0, \dots, p_N\}$, where $p_i = \gamma(i/N)$. Finally, let $\gamma_{\mathbf{p}}$ be the polygonal curve obtained by joining successive points, p_i, p_{i+1} , by the segments of the straight lines they define. We let

$$|\mathbf{p}| = \max\{|p_{i+1} - p_i| : i = 0, 1, \dots, N - 1\},$$

and ω be a C^0 1-form defined in a neighborhood of γ . Prove that, as $|\mathbf{p}| \rightarrow 0$,

$$\int_{\mathbf{p}} \omega \quad \text{converges to} \quad \int_{\gamma} \omega. \quad (5)$$

3. Suppose that S_1 is the unit circle in the plane, centered at $(0, 0)$, which we approximate by a polygonal “stair case”-curve, γ_N obtained as follows: let $p_j = e^{\frac{2\pi ij}{N}} = (x_j, y_j)$, $j = 0, \dots, N$. We join p_j to p_{j+1} by the arc:

$$c(t) = \begin{cases} (1 - 2t)(x_j, y_j) + 2t(x_j, y_{j+1}) & \text{for } t \in [0, \frac{1}{2}] \\ (2 - 2t)(x_j, y_{j+1}) + (1 - 2t)(x_{j+1}, y_{j+1}) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases} \quad (6)$$

Show that

$$\lim_{N \rightarrow \infty} \int_{\gamma_N} ds_N = 8 \quad \text{and} \quad \lim_{N \rightarrow \infty} \int_{\gamma_N} xdy = \pi, \quad (7)$$

where ds_N is arclength along the curve γ_N , and xdy is the 1-form.

4. Verify the chain rule for complex coordinates, e.g. z, \bar{z}, w, \bar{w} : Suppose that $f(z, \bar{z}) : U \rightarrow V$ and $g(w, \bar{w}) : V \rightarrow \mathbb{C}$, are differentiable in the real sense, i.e. as maps from subsets of \mathbb{R}^2 to \mathbb{R}^2 . As noted in class we can think of f and g as functions of the complex variables. If we define $h = g \circ f$, then show that

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial w} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{w}} \frac{\partial \bar{f}}{\partial z}; \quad (8)$$

and

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial w} \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}} \frac{\partial \bar{f}}{\partial \bar{z}}. \quad (9)$$

Recall that $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Let f be analytic in a neighborhood of z . Show that the real derivative of f , $Df(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is invertible at $z = x + iy$ if and only if $f'(z) \neq 0$.

5. If we define $\log z = \log r + i\theta$, at $z = r \cos \theta + ir \sin \theta$, then show that $\log z$ is an analytic function in the set

$$\{z \in \mathbb{C} : r \neq 0 \text{ and } -\pi < \theta < \pi\}.$$

Use this to prove that if f is analytic in $B_R(0)$ and $|f(z)|$ is constant then f is constant.

6. Suppose that f is an analytic function defined in U and that $c : [0, 1] \rightarrow U$ is a C^1 -curve. The composition $t \rightarrow f(c(t))$ can be thought of as a map h from $[0, 1]$ into \mathbb{R}^2 . Show that the first derivative of this map can be computed using the chain rule:

$$Dh(t) = [\operatorname{Re}(f'(c(t))c'(t)), \operatorname{Im}(f'(c(t))c'(t))]. \quad (10)$$

Here we think of c as a complex valued function $c(t) = c_1(t) + ic_2(t)$, with c_1, c_2 real valued functions. Note: This does not follow from (3) as c is not an analytic map (or even defined in an open subset of \mathbb{C}).

7. Show that

$$4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z = \Delta, \quad (11)$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator.

- (a) Show that if $f = u + iv$ is an analytic function in an open subset of \mathbb{C} , then u and v are harmonic, that is

$$\Delta u = \Delta v = 0. \quad (12)$$

(b) Suppose that U is a harmonic function in $B_1(0)$. Show that there is a function V that satisfies the system of equations:

$$V_x = -U_y \text{ and } V_y = U_x. \quad (13)$$

Conclude that $U + iV$ is analytic in $B_1(0)$. If $U = x^5 - 10x^3y^2 + 5xy^4$ what is V ?

8. Define a function on \mathbb{R} by setting

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases} \quad (14)$$

Prove that f is infinitely differentiable. Can f be the restriction, to the real axis, of an analytic function defined in a neighborhood of $(0, 0)$?