

AMCS 608

Problem set 5 due October 20, 2009

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**Reading:** There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

**Standard problems:** The following problems should be done, but do not have to be handed in.

1. We can define polar coordinates in the complex plane by setting

$$x = r \cos \theta \quad y = r \sin \theta. \quad (1)$$

Show that in polar coordinates the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}. \quad (2)$$

2. Show that if  $f$  is analytic in  $B_R(0)$  and  $\operatorname{Re} f$ , or  $\operatorname{Im} f$  is a constant, then  $f$  is constant.
3. Suppose that  $g$  is an analytic function defined in  $U \subset \mathbb{C}$  and  $f$  is an analytic defined in an open set containing  $g(U)$ . Show that  $f \circ g$  is an analytic function, using just the definition of complex derivative. Show that the chain rule is valid:

$$(f \circ g)'(w) = f'(g(w))g'(w). \quad (3)$$

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Verify the chain rule for complex coordinates, e.g.  $z, \bar{z}, w, \bar{w}$ : Suppose that  $f(z, \bar{z}) : U \rightarrow V$  and  $g(w, \bar{w}) : V \rightarrow \mathbb{C}$ , are differentiable in the real sense, i.e. as maps from subsets of  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . As noted in class we can think of  $f$  and  $g$  as functions of the complex variables. If we define  $h = g \circ f$ , then show that

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial w} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{w}} \frac{\partial \bar{f}}{\partial z}; \quad (4)$$

and

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial w} \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}} \frac{\partial \bar{f}}{\partial \bar{z}}. \quad (5)$$

Recall that  $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$  and  $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ . Let  $f$  be analytic in a neighborhood of  $z$ . Show that the real derivative of  $f$ ,  $Df(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is invertible at  $z = x + iy$  if and only if  $f'(z) \neq 0$ .

2. If we define  $\log z = \log r + i\theta$ , at  $z = r \cos \theta + ir \sin \theta$ , then show that  $\log z$  is an analytic function in the set

$$\{z \in \mathbb{C} : r \neq 0 \text{ and } -\pi < \theta < \pi\}.$$

Use this and the polar form of the Cauchy-Riemann to prove that if  $f$  is analytic in  $B_R(0)$  and  $|f(z)|$  is constant then  $f$  is constant.

3. Suppose that  $f$  is an analytic function defined in  $U$  and that  $c : [0, 1] \rightarrow U$  is a  $C^1$ -curve. The composition  $t \rightarrow f(c(t))$  can be thought of as a map  $h$  from  $[0, 1]$  into  $\mathbb{R}^2$ . Show that the first derivative of this map can be computed using the chain rule:

$$Dh(t) = [\operatorname{Re}(f'(c(t))c'(t)), \operatorname{Im}(f'(c(t))c'(t))]. \quad (6)$$

Here we think of  $c$  as a complex valued function  $c(t) = c_1(t) + ic_2(t)$ , with  $c_1, c_2$  real valued functions. Note: This does not follow from (3) as  $c$  is not an analytic map (or even defined in an open subset of  $\mathbb{C}$ ).

4. Using the definition of the complex contour integral evaluate the following integrals over  $\gamma$ , the semi-circle  $\{z : |z| = 1, \operatorname{Im} z > 0\}$ .

(a)  $\int_{\gamma} \frac{dz}{z^n}$  where  $n \in \mathbb{N}$ .

(b)  $\int_{\gamma} \bar{z}^n dz$ , where  $n \in \mathbb{Z}$ .

(c)  $\int_{\gamma} z^n \bar{z}^n dz$ , where  $n \in \mathbb{Z}$ .

5. Show that

$$4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z = \Delta, \quad (7)$$

where  $\Delta = \partial_x^2 + \partial_y^2$  is the Laplace operator.

(a) Show that if  $f = u + iv$  is an analytic function in an open subset of  $\mathbb{C}$ , then  $u$  and  $v$  are harmonic, that is

$$\Delta u = \Delta v = 0. \quad (8)$$

(b) Suppose that  $U$  is a harmonic function in  $B_1(0)$ . Show that there is a function  $V$  that satisfies the system of equations:

$$V_x = -U_y \text{ and } V_y = U_x. \quad (9)$$

Conclude that  $U + iV$  is analytic in  $B_1(0)$ . If  $U = x^5 - 10x^3y^2 + 5xy^4$  what is  $V$ ?

6. Let  $f(x, y) = \sqrt{|x||y|}$ , for all  $(x, y) \in \mathbb{R}^2$ . Show that  $f$  satisfies the Cauchy-Riemann equation at  $(0, 0)$ , but does not have a complex derivative at this point.

7. Prove the following:

(a) The power series  $\sum_{n=1}^{\infty} nz^n$  converges for any  $z$  with  $|z| < 1$ , but does not converge for any point where  $|z| = 1$ .

(b) The power series  $\sum_{n=1}^{\infty} z^n/n^2$  converges for any  $z$  with  $|z| \leq 1$ .

(c) The power series  $\sum_{n=1}^{\infty} z^n/n$  converges for any  $z$  with  $|z| \leq 1$ , except  $z = 1$ .

8. Define a function on  $\mathbb{R}$  by setting

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases} \quad (10)$$

Prove that  $f$  is infinitely differentiable.