

AMCS 608

Problem set 6 due November 2, 2010

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**Reading:** There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

**Standard problems:** The following problems should be done, but do not have to be handed in.

1. Show that there does not exist an analytic function in  $D_1(0)$ , which extends continuously to  $\{z : |z| = 1\}$ , so that  $f(z) = 1/z$  on the unit circle.
2. Let  $\{w_1, \dots, w_m\}$  be points in the unit circle. Show that there exists a point,  $z$  on the unit circle where

$$\prod_{j=1}^m |z - w_j| > 1. \quad (1)$$

Conclude that there are also points on the unit circle where

$$\prod_{j=1}^m |z - w_j| = 1. \quad (2)$$

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. (a) Suppose that  $f : D_1(0) \rightarrow \mathbb{C}$  is an analytic map and  $f'(0) \neq 0$ . Show that there is an  $r > 0$  and an *analytic* map  $g : D_r(f(0)) \rightarrow D_1(0)$ , satisfying  $g(f(0)) = 0$ ,  $f(g(w)) = w$ , for  $w \in D_r(f(0))$ , and  $g(f(z)) = z$ , for  $z$  closed enough to 0. Be sure to show that  $g$  has a complex derivative throughout  $D_r(f(0))$ . Hint: The inverse function theorem for maps from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .
- (b) Show that for any  $n \in \mathbb{N}$  and  $w_0 \in \mathbb{C} \setminus \{0\}$  there is an analytic  $n$ th root function  $g_n(w)$  defined in a neighborhood  $U$  of  $w_0$ , so that, for  $w \in U$ , we have  $(g_n(w))^n = w$ . Explain why, if  $n > 1$ , no such function can be analytic in a neighborhood of 0.

- (c) Show that there is a neighborhood  $U$  of any point  $w_0 \in \mathbb{C} \setminus \{0\}$  in which an analytic function  $l(w)$  is defined that satisfies

$$e^{l(w)} = w, \quad (3)$$

for  $w \in U$ . What is the real part of  $l$ ? This is a branch of the log. For this problem we can take  $e^z$  to be the entire function defined by the power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \quad (4)$$

You need to prove that  $e^z$  satisfies the hypotheses in part (a).

- Let  $f$  be a non-constant analytic function defined in a neighborhood of the closed unit disk. Suppose that  $|f(z)| = 1$  where  $|z| = 1$ . Show that  $\theta \rightarrow f(e^{i\theta})$  goes counterclockwise around the unit disk, and makes at least one full rotation. Hint: In a neighborhood of any point on the unit disk  $\log f(z) = \log |f(z)| + i \arg f(z)$  is an analytic function. Use the maximum principle.
- Suppose that  $\alpha \in D_1(0)$ , and  $\theta \in \mathbb{R}$ . Show that

$$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad (5)$$

is a 1-1, onto analytic map of  $D_1(0)$  to itself. Show that every 1-1, onto analytic self map of  $D_1(0)$  is of this form. Show that  $g(z) = i(1+z)/(1-z)$  is a 1-1, onto analytic map from the unit disk to  $H_+ = \{z : \text{Im } z > 0\}$ . Use this map and the first part of the problem to find all the 1-1, onto analytic maps of  $H_+$  to itself.

- Suppose that  $f$  is analytic in  $D_{R_0}(0)$ . Show that whenever  $0 < R < R_0$  and  $|z| < R$ , then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) \text{Re} \left( \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right) d\theta. \quad (6)$$

Show that

$$\text{Re} \left( \frac{Re^{i\theta} + r}{Re^{i\theta} - r} \right) = \frac{R^2 - r^2}{R^2 - 2rR \cos \theta + r^2}. \quad (7)$$

Finally, if  $f = u + iv$ , and  $f(0)$  is real, then show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left( \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right) d\theta. \quad (8)$$

5. Let  $U \subset \mathbb{C}$  be an open set and define the norms

$$\|f\|_{L^2(U)} = \sqrt{\iint_U |f(z)|^2 dx dy}, \quad (9)$$

and

$$\|f\|_{L^\infty(U)} = \sup_{z \in U} |f(z)|. \quad (10)$$

Suppose that  $f$  is holomorphic in  $D_1(0)$  show that, for each  $0 < s < r < 1$ , there is a constant  $C$  (depending on  $r, s$ , but not on  $f$ ) so that

$$\|f\|_{L^\infty(D_s(0))} \leq C \|f\|_{L^2(D_r(0))}. \quad (11)$$

Conclude that if  $\langle f_n \rangle$  is a sequence of analytic functions for which there is a function  $f \in L^2(D_1(0))$ , such that

$$\lim_{n \rightarrow \infty} \|f - f_n\|_{L^2(D_1(0))} = 0, \quad (12)$$

then  $f$  is also analytic in  $D_1(0)$ .