AMCS 608

Problem set 7 due November 3, 2009

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Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

Standard problems: The following problems should be done, but do not have to be handed in.

1. For $0 \leq r < R$ show that the annular region:

$$A_{rR} = \{z : r < |z| < R\} \tag{1}$$

   is not simply connected.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Use Liouville's theorem to give an entirely analytic proof that a polynomial in z, of degree $n > 0$, has at least one root, and therefore has exactly n roots, counted with multiplicity.

2. Suppose that f is an analytic function in a neighborhood of the unit disk, except for a pole at z_0, where $|z_0| = 1$. Show that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2}$$

   in the unit disk, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0. \tag{3}$$

3. Suppose that h is a continuous function on $\mathbb{R}$ with support in the finite interval $[-M, M]$, and define

$$g(z) = \frac{1}{2\pi i} \int_{-M}^{M} \frac{h(x)dx}{x - z} \tag{4}$$

(a) Prove that g is an analytic function in $\mathbb{C} \setminus [-M, M]$, which tends to zero as $|z|$ tends to infinity.

(b) Prove that the jump in g across $[-M, M]$ satisfies

$$\lim_{\epsilon \to 0^+} [g(x + i\epsilon) - g(x - i\epsilon)] = h(x). \tag{5}$$

(c) Show that if h is continously differentiable on $(-M, M)$, and $h'(x)$ extends continuously to $[-M, M]$, then $g(x + i\epsilon)$ and $g(x - i\epsilon)$ converge uniformly to limits, $g_+(x)$, $g_-(x)$, respectively, as $\epsilon \to 0^+$. In this case, show that $g(z)$ is the *unique* function, which is holomorphic in $\mathbb{C} \setminus [-M, M]$, vanishes as $|z| \to \infty$, has limits g_+, and g_- as $[-M, M]$ is approached from above and below, which satisfies the jump condition:

$$g_+(x) - g_-(x) = h(x) \text{ for } x \in [-M, M]. \tag{6}$$

Hint: Consider

$$g(y \pm i\epsilon) = \frac{1}{2\pi i} \int_{-M}^{M} \frac{(h(x) - h(y))dx}{x - (y \pm i\epsilon)} + \frac{1}{2\pi i} \int_{-M}^{M} \frac{h(y)dx}{x - (y \pm i\epsilon)} \tag{7}$$

4. For each $z \in \mathbb{C}$ the function

$$g(z; t) = \exp\left(\frac{z}{2}(t - t^{-1})\right) \tag{8}$$

is an analytic function of $t \in \mathbb{C} \setminus \{0\}$ and therefore has a Laurent expansion:

$$g(z; t) = \sum_{n=-\infty}^{\infty} J_n(z)t^n, \tag{9}$$

convergent in this set.

(a) Show that

$$J_n(z) = \frac{1}{2\pi i} \int_{\{t:\,|t|=1\}} \frac{g(z; t)dt}{t^{n+1}}. \tag{10}$$

Conclude that $J_n(z)$ is an entire function of z for every $n \in \mathbb{Z}$, and that $J_{-n}(z) = J_n(-z)$.

(b) Use this contour integral to estimate $J_n(z)$. Can you show that there is a constant C so that

$$|J_n(z)| \leq C\sqrt{|n|+1}\frac{\left(\frac{|z|}{2}\right)^{|n|} e^{|z|}}{|n|!}? \tag{11}$$

(c) Use the generating function g to show that the following identities hold:

$$\begin{aligned} J_n'(z) &= \frac{1}{2}(J_{n-1}(z) - J_{n+1}(z)) \\ J_n(z) &= \frac{z}{2n}(J_{n-1}(z) + J_{n+1}(z)) \text{ for } n \neq 0 \\ J_n''(z) &+ \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0. \end{aligned} \tag{12}$$

5. Prove that if f is an analytic function in all of $\mathbb{C}$, except for poles, and f has, at worst, a pole at infinity, then there polynomials p and q so that

$$f(z) = \frac{p(z)}{q(z)}. \tag{13}$$

6. Suppose that u is a C^2 harmonic function in a simply connected set U, that is $\Delta u = \partial_x^2 u + \partial_y^2 u = 0$. Show that there is a C^1 function v defined in U so that

$$\partial_x v = -\partial_y u \text{ and } \partial_y v = \partial_x u. \tag{14}$$

Show that $u + iv$ is an analytic function. Find an example to show that if U is not simply connected, then v may not be globally defined in U.