

AMCS 608

Problem set 8 due November 10, 2009

Dr. Epstein

Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari. Conformal Mapping is an especially good reference for material on harmonic functions.

Standard problems: The following problems should be done, but do not have to be handed in.

1. Evaluate the integral $\int_0^{\infty} \frac{dx}{1+x^4}$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Evaluate the following integrals:

(a) $\int_0^{\infty} \frac{x^{\alpha-1} dx}{(x+\beta)(x+\gamma)}$, where $0 < \alpha < 1$, and $\beta, \gamma > 0$.

(b) $\int_0^{\infty} \frac{\cos x dx}{a^2+x^2}$, for $a > 0$.

(c) $\int_{-\infty}^{\infty} \frac{x \sin x dx}{a^2+x^2}$, for $a > 0$. Note that this integral is not absolutely convergent, so you need to explain the meaning of this integral as a limit of integrals over finite intervals.

(d) Show that, for $n \in \mathbb{N}$, we have

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi.$$

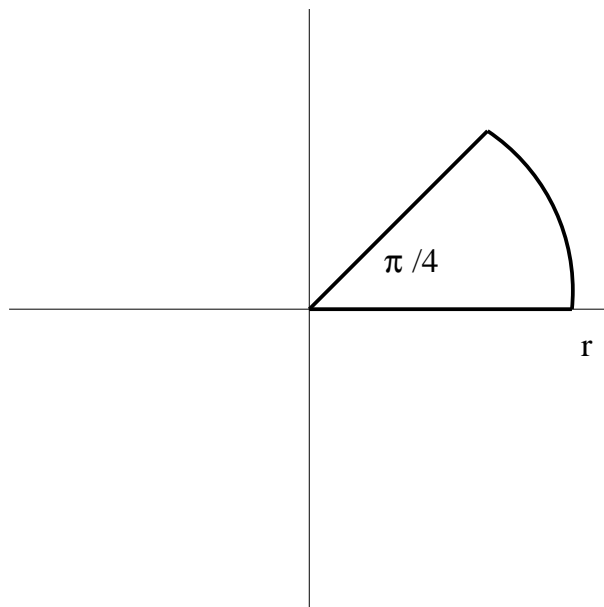


Figure 1. The integration contour Γ_r .

2. Compute the improper Riemann integrals:

$$\int_0^{\infty} \cos(x^2) dx = \lim_{r \rightarrow \infty} \int_0^r \cos(x^2) dx, \quad \int_0^{\infty} \sin(x^2) dx = \lim_{r \rightarrow \infty} \int_0^r \sin(x^2) dx, \quad (1)$$

by evaluating the contour integral,

$$\lim_{r \rightarrow \infty} \int_{\Gamma_r} e^{-z^2} dz, \quad (2)$$

where Γ_r is the contour shown in Figure 1.

3. Let f be a 1-1 analytic function defined in $D_1(0)$, with $f(0) = w_0$. As shown in class, the inverse of f for w a neighborhood of w_0 is given by

$$g(w) = \frac{1}{2\pi i} \int_{|z|=1} \frac{zf'(z)}{f(z) - w} dz. \quad (3)$$

Use the fact that

$$\frac{1}{f(z) - w} = \frac{1}{f(z) - w_0 - (w_0 - w)}, \quad (4)$$

to derive a formula, in terms of f , for the Taylor coefficients of g at w_0 .

4. Show that if $|a| < 1$, then

$$\int_0^{2\pi} \log |1 - ae^{i\theta}| d\theta = 0. \quad (5)$$

5. Let $\{z_1, \dots, z_n\}$ be points lying inside the simple closed curve γ and define

$$p(z) = \prod_{j=1}^n (z - z_j). \quad (6)$$

If f is analytic inside of γ and continuous up to γ , then show that

$$P(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) p(\zeta) - p(z)}{p(\zeta) \zeta - z} d\zeta \quad (7)$$

is a polynomial of degree $n - 1$, which satisfies:

$$P(z_j) = f(z_j) \text{ for } j = 1, \dots, n. \quad (8)$$

6. Show that, if $a > 1$, then the equation

$$ze^{a-z} = 1 \quad (9)$$

has precisely one root in the unit disk $\{z : |z| \leq 1\}$. Explain why this root is necessarily a positive real number.

7. Prove that the sequence of entire functions, $f_n(z) = (1 + \frac{z}{n})^n$ converges locally uniformly to $f(z) = e^z$. Using this fact, prove that $f(z) = 0$ has no solution.