AMCS 608

Problem set 9 due November 23, 2010

Dr. Epstein

Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari. Conformal Mapping is an especially good reference for material on harmonic functions.

Standard problems, do not hand in:

Suppose that D is a bounded connected region in the plane, with a C^1 boundary, and u, v are twice continuously differentiable functions in D, whose first derivatives have continuous extensions to $\overline{D}$. We let $\boldsymbol{n}$ denote the outer unit normal vector along bD, and define the normal derivative of u along the boundary:

$$\frac{\partial u}{\partial \boldsymbol{n}} = \langle \nabla u, \boldsymbol{n} \rangle. \tag{1}$$

1. Show that *Stokes' Theorem* (for 1-forms) implies that

$$\int_D [u_x v_x + u_y v_y] dxdy + \int_D u \Delta v dxdy = \int_{bD} u \frac{\partial v}{\partial \boldsymbol{n}} ds. \tag{2}$$

   Here ds denotes arclength measure along bD.

2. Use this formula to deduce that if u is also harmonic in D, then

$$\int_{bD} \frac{\partial u}{\partial \boldsymbol{n}} ds = 0. \tag{3}$$

3. Show that equation (2) implies that

$$\int_D [u \Delta v - v \Delta u] dxdy = \int_{bD} [u \frac{\partial v}{\partial \boldsymbol{n}} - v \frac{\partial u}{\partial \boldsymbol{n}}] ds. \tag{4}$$

4. Show that if u is harmonic on $D_R(0)$ and continuous on $\overline{D_R(0)}$, then, for $z \in D_R(0)$ we have

$$u(z, \bar{z}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - |z|^2) u(Re^{i\theta}, Re^{-i\theta})}{|Re^{i\theta} - z|^2} d\theta. \tag{5}$$

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Using (4) show that if u is a C^2-function with compact support then
$$\int\limits_{\mathbb{C}} \log(x^2+y^2)\Delta u(x,y)dxdy = 4\pi u(0). \tag{6}$$
If u is a compactly supported, C^2-function, then show that the function
$$U(x,y) = \frac{1}{4\pi}\int\limits_{\mathbb{C}} \log(x'^2+y'^2)u(x-x',y-y')dx'dy' \tag{7}$$
is twice differentiable and satisfies
$$\Delta U = u. \tag{8}$$
Hint: Be careful because $\log(x^2+y^2)$ is singular at $(0,0)$.

2. Suppose that u is a harmonic function defined in a simply connected domain, D with C^1-boundary, and let v denote a harmonic conjugate to u. Suppose that the first derivatives of u and v extend continuously to the bD. For a differentiable function f defined along bD let $\frac{\partial f}{\partial s}$ denote the derivative of f with respect to arclength along bD. If $\boldsymbol{t}$ is the unit tangent vector to bD, oriented in the positive direction, then the tangential derivative is:
$$\frac{\partial(v\restriction_{bD})}{\partial s} = \langle\nabla v, \boldsymbol{t}\rangle\restriction_{bD}. \tag{9}$$
   (a) Show that along bD we have the relation:
$$\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial v}{\partial s}. \tag{10}$$
   (b) Let $g(s)$ be a continuous function defined along $bD_1(0)$ with s the arclength parameter and
$$\int\limits_{bD_1(0)} g(s)ds = 0. \tag{11}$$
   Explain how to use (10) to prove that there is a harmonic function u defined in $D_1(0)$ such that
$$\frac{\partial u}{\partial \boldsymbol{n}}(s) = g(s). \tag{12}$$
   Hint: The function $G(s) = \int\limits_{s_0}^{s} g(\sigma)d\sigma$ is a continous function on bD_1.

3. A polynomial $p(x, y)$ is homogeneous of degree k if $\lambda \in (0, \infty)$,

$$p(\lambda x, \lambda y) = \lambda^k p(x, y) \text{ for all } (x, y) \in \mathbb{R}^2. \tag{13}$$

Any polynomial p of degree can be written as sum of homogeneous polynomials. Prove that a polynomial in (x, y) is harmonic if and only if each homogeneous part is harmonic. For each $n \in \mathbb{N}$ find a basis for the two dimensional, real vector space, $\mathcal{H}_n$, of homogeneous, harmonic polynomials of degree n. You must prove that $\dim \mathcal{H}_n = 2$. Suppose that p is a homogeneous polynomial of degree n, show that there are harmonic polynomials $\{h_j\}$ of degrees $j \in \{n, n-2, \dots, 2, 0\}$, if n is even, and $j \in \{n, n-2, \dots, 3, 1\}$, if n is odd, so that

$$p(x, y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} r^{2j} h_{n-2j}(x, y), \tag{14}$$

where $r^2 = x^2 + y^2$.

4. Show that if u is harmonic on $D_R(0)$ and continuous on $\overline{D_R}(0)$, then equation (5) implies that:

$$\partial_z u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-i\theta} u(Re^{i\theta}, Re^{-i\theta})}{R} d\theta. \tag{15}$$

Prove that a bounded harmonic function defined in the whole complex plane is constant. Let u be a bounded harmonic function defined in a domain D. Show that the gradient of u satisfies the estimate

$$|\nabla u(x, y)| \leq \frac{2M}{\text{dist}((x, y), D^c)}, \tag{16}$$

provided $|u(z)| \leq M$ in D.

5. Let u be a continuous function defined in a connected open set D. Let $z \in D$, and suppose that $r_z = \text{dist}(z, D^c)$. We say that u satisfies the mean value property in D if, for every $z \in D$ and $r < r_z$ we have that

$$u(z, \bar{z}) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}, \bar{z} + re^{-i\theta}) d\theta. \tag{17}$$

Prove that a C^2-function that satisfies the mean value property is harmonic. It is actually true that a C^0-function satisfying the mean value property is harmonic. Can you prove it?

6. Let $< u_n >$ be a sequence of harmonic functions defined in a connected open set D. Suppose that $< u_n >$ converges locally uniformly to a function u. Prove that the limit is also harmonic. Do **not** use the harmonic conjugate. Hint: This is a local property.