

AMCS 609

Problem set 1 due January 27, 2009

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Reading: There are many excellent references for this material; two I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, Appendix A, and *Asymptotic Expansions* by A. Erdélyi.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Prove that

$$F(s) = \lim_{R_1, R_2 \rightarrow \infty} \int_{-R_1}^{R_2} e^{-sx^2} dx \quad (1)$$

defines a smooth function in $\{s : \operatorname{Re} s \geq 0\} \setminus \{0\}$.

2. How does $\frac{(2n)!}{(n!)^2}$ behave as $n \rightarrow \infty$? What is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!z^n}{(n!)^2} ? \quad (2)$$

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that f is a bounded, continuous function in $[0, A]$. Show that

$$\lim_{n \rightarrow \infty} \frac{(n+1) \int_0^A t^n f(t) dt}{A^{n+1}} = f(A). \quad (3)$$

2. Let $\varphi \in \mathcal{C}_c^\infty(-1, 1)$ be a non-negative even function, which equals 1 in an interval $[-\epsilon, \epsilon]$. Prove that for s , with $\operatorname{Re} s \geq 0$, and any $N > 0$, we have

$$\int_{-1}^1 e^{-sx^2} \varphi(x) dx = \sqrt{\frac{\pi}{s}} + O(|s|^{-N}). \quad (4)$$

3. Suppose that ψ is a smooth function with compact support, show that for any $N \in \mathbb{N}$:

$$\int_{-\infty}^{\infty} e^{ix\xi} \psi(x) dx = O\left(\frac{1}{|\xi|^N}\right), \quad (5)$$

as $|\xi|$ tends to infinity, with $|\operatorname{Im} \xi| < M$.

4. Suppose that ψ is a smooth function with compact support in $(-1, 1)$, and define

$$F(s) = \int_{-1}^1 e^{-sx^4} \psi(x) dx. \quad (6)$$

Show that there is a universal constant C so that, as $s \rightarrow \infty$ on the real axis, we have

$$F(s) = \frac{C\psi(0)}{s^{\frac{1}{4}}} \left[1 + O(s^{-\frac{1}{2}})\right]. \quad (7)$$

5. Prove that

$$\int_0^{\infty} \frac{e^{-xt}}{1+t^2} dt \sim \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \dots \quad (8)$$

That is, the right hand series is an asymptotic expansion for the function defined in the left as $x \rightarrow \infty$. Show that the series on the right does not converge for any finite value of x .

6. Let $\nu \in [0, \infty)$, and $J_\nu(x)$ denote the order ν -Bessel function. Show that there is an N_ν so that, if $n > N_\nu$, then the equation

$$J_\nu(s) = 0, \quad (9)$$

has unique solutions

$$s_n \in \left[n\pi + \frac{\pi\nu}{2} + \frac{\pi}{4}, (n+1)\pi + \frac{\pi\nu}{2} + \frac{\pi}{4}\right]. \quad (10)$$

Prove, using the asymptotic expansion, that these roots are all simple.

Use the MATLAB function “BesselJ” to graph J_ν for several values of ν and s in intervals of the form $[0, S]$. Can you guess the approximate size of N_ν ?