AMCS/MATH 609
Problem set 1 due January 27, 2015
Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi. **Standard problems:** The solutions to the following problems do not need to be handed in.

1. Let f be a measurable function defined in $\mathbb{R}^d$ and define
$$\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^d\}. \tag{1}$$
Show that Γ_f is a measurable set, and that $m(\Gamma_f) = 0$.

2. Show that there does *not* exist a function $I \in L^1(\mathbb{R}^d)$ so that, for every $f \in L^1(\mathbb{R}^d)$,
$$f * I = f. \tag{2}$$

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that F is a closed set in $\mathbb{R}$ whose complement has finite measure. Let $\delta(x)$ denote the distance to F :
$$\delta(x) = \inf\{|x - y| : y \in F\}, \tag{3}$$
and let
$$I(x) = \int\limits_{\mathbb{R}} \frac{\delta(y)dy}{|x - y|^2} \tag{4}$$
   (a) Prove that δ is continuous, by showing that it satisfies the Lipschitz condition:
$$|\delta(x) - \delta(y)| \leq |x - y|. \tag{5}$$
   (b) Show that $I(x) = \infty$ for each $x \notin F$.
   (c) Show that $I(x) < \infty$ for a.e. $x \in F$. [Hint: Investigate $\int_F \delta(x)dx$.]

2. Let f be a measurable finite valued function defined on $[0, 1]$, and suppose that $|f(x) - f(y)|$ is integrable over $[0, 1] \times [0, 1]$. Show that $f(x)$ is integrable over $[0, 1]$.

3. Suppose that f and g are measurable functions on $\mathbb{R}^d$.

   (a) Prove that $f(x - y)g(y)$ is measurable on $\mathbb{R}^{2d}$.

   (b) Show that if f and g are integrable on $\mathbb{R}^d$, then $f(x - y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

   (c) Show that the convolution $f * g(x)$,

   $$f * g(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy \tag{6}$$

   is well defined for a.e. $x \in \mathbb{R}^d$.

   (d) Show that if f and g are integrable on $\mathbb{R}^d$, then

   $$\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}, \tag{7}$$

   with equality if f and g are non-negative.

   (e) The Fourier transform of an L^1-function is defined by

   $$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}dx. \tag{8}$$

   Show that $\hat{f}(\xi)$ is a bounded and continuous of ξ. If if f and g are integrable, then show that, for each ξ we have

   $$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi). \tag{9}$$

4. Let $K_\delta(x) = e^{-\pi|x|^2/\delta}\delta^{-\frac{d}{2}}$, and for $\epsilon > 0$ define

   $$f_\epsilon(x) = \int_0^\infty K_\delta(x)e^{-\pi\delta}\delta^{\epsilon-1}d\delta. \tag{10}$$

   Use Fubini's theorem to show that $f_\epsilon \in L^1(\mathbb{R}^d)$ and show that

   $$\hat{f}_\epsilon(\xi) = \int_0^\infty e^{-\pi\delta|\xi|^2}e^{-\pi\delta}\delta^{\epsilon-1}d\delta. \tag{11}$$

Finally show that there is a positive constant, C_ϵ, so that

$$\hat{f}_\epsilon(\xi) = \frac{C_\epsilon}{(1+|\xi|^2)^\epsilon} \tag{12}$$

5. Suppose that f is a non-trivial element of $L^1(\mathbb{R}^d)$, and let f^* be its maximal function. Show that there is a constant $c > 0$ so that

$$f^*(x) \geq \frac{c}{|x|^d} \text{ for all } |x| \geq 1, \tag{13}$$

and hence f^* is not integrable on $\mathbb{R}^d$.

6. Define the function on $\mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{|x|(\log|x|)^2} & \text{for } |x| \leq \frac{1}{2} \\ 0 & \text{for } |x| > \frac{1}{2}. \end{cases} \tag{14}$$

Show that f is integrable, but there is a positive constant c so that

$$f^*(x) \geq \frac{c}{|x||\log|x||} \text{ for } |x| \leq \frac{1}{2}, \tag{15}$$

and therefore f^* is not locally integrable.