

AMCS 609

Problem set 1 due January 27, 2011

Dr. Epstein

**Reading:** There are many excellent references for this material; two I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, and *Asymptotic Expansions* by A. Erdelyi.

**Standard problems:** The following problems should be done, but do not have to be handed in.

1. Suppose that  $\psi$  is a smooth function with compact support, show that for any  $N \in \mathbb{N}$ :

$$\int_{-\infty}^{\infty} e^{ix\zeta} \psi(x) dx = O\left(\frac{1}{|\zeta|^N}\right), \quad (1)$$

as  $|\zeta|$  tends to infinity, with  $|\operatorname{Im} \zeta| < M$ .

2. Show that there is a constant  $C$ , so for any  $\epsilon > 0$  we have the estimate

$$\left| \int_{-\epsilon}^{\epsilon} e^{-\lambda x^2} dx - \sqrt{\frac{\pi}{\lambda}} \right| \leq C \frac{e^{-\lambda \epsilon^2}}{\sqrt{\lambda}}, \quad (2)$$

as  $\lambda \rightarrow \infty$ .

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Prove that

$$F(s) = \lim_{R \rightarrow \infty} \int_0^R e^{-sx^2} dx \quad (3)$$

defines a smooth function in  $\{s : \operatorname{Re} s \geq 0\} \setminus \{0\}$ . You need to show that the limit exists for  $s$  in the closed half plane, and that all derivatives have continuous extensions up to  $\operatorname{Re} s = 0$ . Hint: integrate by parts.

2. Suppose that  $f$  is a bounded, continuous function in  $[0, A]$ . Show that

$$\lim_{n \rightarrow \infty} \frac{(n+1) \int_0^A t^n f(t) dt}{A^{n+1}} = f(A). \quad (4)$$

3. Let  $\varphi \in \mathcal{C}_c^\infty(-1, 1)$  be a non-negative even function, which equals 1 in an interval  $[-\epsilon, \epsilon]$ . Prove that for  $s$ , with  $\operatorname{Re} s \geq 0$ , and any  $N > 0$ , we have

$$\int_{-1}^1 e^{-sx^2} \varphi(x) dx = \sqrt{\frac{\pi}{s}} + O(|s|^{-N}). \quad (5)$$

4. Suppose that  $\psi$  is a smooth function with compact support in  $(-1, 1)$ , and define

$$F(s) = \int_{-1}^1 e^{-sx^4} \psi(x) dx. \quad (6)$$

Show that there is a universal constant  $C$  so that, as  $s \rightarrow \infty$  on the real axis, we have

$$F(s) = \frac{C \psi(0)}{s^{\frac{1}{4}}} \left[ 1 + O(s^{-\frac{1}{2}}) \right]. \quad (7)$$

5. Starting from the definition

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(x \sin \theta) d\theta, \quad (8)$$

show that

$$J_0(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) + O(x^{-1}). \quad (9)$$

6. Prove that

$$\int_0^\infty \frac{e^{-xt} dt}{1+t^2} \sim \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \dots \quad (10)$$

That is, the right hand series is an asymptotic expansion for the function defined in the left as  $x \rightarrow \infty$ . Show that the series on the right does not converge for any finite value of  $x$ .

Extra Credit: Let  $\nu \in \mathbb{N}$ , and  $J_\nu(x)$  denote the order  $\nu$ -Bessel function, which is given by

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \sin^{2\nu} t e^{ix \cos t} dt. \quad (11)$$

Prove the asymptotic expansion for  $J_\nu(x)$  :

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O(x^{-\frac{3}{2}}). \quad (12)$$

Show that there is an  $N_\nu$  so that, if  $n > N_\nu$ , then the equation

$$J_\nu(x) = 0, \quad (13)$$

has unique solutions

$$x_n \in \left[n\pi + \frac{\pi\nu}{2} + \frac{\pi}{4}, (n+1)\pi + \frac{\pi\nu}{2} + \frac{\pi}{4}\right]. \quad (14)$$

Can you explain why these zeros are simple?

Use the MATLAB function “BesselJ” to graph  $J_\nu$  for several values of  $\nu$  and  $s$  in intervals of the form  $[0, S]$ . Can you guess the approximate size of  $N_\nu$ ?