

AMCS 609

Problem set 10 due April 19, 2011

Dr. Epstein

Reading: Read Chapters 21 and 22 in Lax, *Functional Analysis*.

Standard problem: The following problem should be done, but does not have to be handed in.

1. Let X be a Banach space, and $K \subset X$ a precompact subset of X . Show that the convex hull of K is also precompact.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that the two definitions of precompact set given on page 233 of Lax are equivalent.
2. Let X be a Banach space and $\{P_N : N \in \mathbb{N}\}$ be a sequence of finite rank operators, which converge strongly to the identity, that is $\lim_{N \rightarrow \infty} P_N x = x$, for every $x \in X$. If $C : X \rightarrow X$ is a compact operator, then prove that $P_N C$ converges to C in the uniform norm. Show that if X is a Hilbert space, then any compact map $C : X \rightarrow X$ is the norm limit of a sequence of finite rank maps.
3. Let $C : X \rightarrow X$ be a compact map, X a Banach space. For $\lambda \in \mathbb{C} \setminus \{0\}$ we let

$$E_\lambda = \{x \in X : Cx = \lambda x\}. \tag{1}$$

- (a) Show that $\dim E_\lambda < \infty$.
 - (b) Show that the set $\{\lambda : \dim E_\lambda > 0\}$, is a discrete subset of $\mathbb{C} \setminus \{0\}$.
4. Suppose that X, U are Banach spaces and $C : X \rightarrow U$ is compact. Prove that if $\langle x_n \rangle$ is sequence in X , which converges weakly to x , then $\langle Cx_n \rangle$ converges in norm to Cx .
 5. Suppose that X is a Hilbert space and $C : X \rightarrow X$ is a compact *self adjoint* operator, that is $\langle Cx, y \rangle = \langle x, Cy \rangle$, for all $x, y \in X$.
 - (a) Prove that for all $x \in X$, the function $F(x) = \langle Cx, x \rangle$ is real valued.

(b) Suppose that for some x , $F(x) > 0$; show that there is unit vector $x_1 \in X$, so that

$$F(x_1) = \sup\{F(x) : x \in X \text{ with } \|x\| = 1\}. \quad (2)$$

(c) Prove that x_1 is an eigenvector of C , that is, there is a real number λ_1 so that $Cx_1 = \lambda_1 x_1$. Give an example of a bounded (though non-compact), self adjoint operator A on a Hilbert space for which this is **not** true.

(d) If we let $X_1 = \{x \in X : \langle x, x_1 \rangle = 0\}$, then C maps X_1 to itself, that is $CX_1 \subset X_1$.

6. Let $k(s, t)$ be a C^1 -function on $[0, 1] \times [0, 1]$. Define the operator K by

$$Kf(s) = \int_0^1 k(s, t)f(t)dt. \quad (3)$$

Show that $K : C^0([0, 1]) \rightarrow C^0([0, 1])$ and $K : L^2([0, 1]) \rightarrow L^2([0, 1])$ are compact operators.