AMCS/MATH 609
Problem set 10 due April 21, 2015
Dr. Epstein

Reading: Read Chapters 15 and 16 in Lax, *Functional Analysis*.

Standard problem: The following problem should be done, but do not have to be handed in.

1. Show that if P is a non-trivial projection, that is $P^2 = P$, but $P \neq 0$, then $|P| \geq 1$.

2. Show that if
$$\int_{-\infty}^{\infty} |\phi(x)| dx < \infty, \tag{1}$$
then the operator
$$K_\phi f(x) = \int_{-\infty}^{\infty} \phi(x-y) f(y) dy \tag{2}$$
is bounded from $L^2(\mathbb{R})$ to itself.

3. Let X be a separable Banach space with $\{x_n\}$ a countable dense subset of the unit ball. We define a map $T : \ell_1 \to X$, by setting:
$$T(\boldsymbol{a}) = \sum_{j=1}^{\infty} a_j x_j. \tag{3}$$
   (a) Prove that T is bounded.
   (b) Prove that T is surjective. Hint: you should find a direct argument.
   (c) Show that X is isomorphic to a quotient space of ℓ_1.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that if $g \in L^1([-\pi, \pi])$, and we define
$$a_n = \int_{-\pi}^{\pi} g(x) e^{-inx} dx, \tag{4}$$

then $\lim_{n\to\pm\infty} a_n = 0$. Hint: Approximate g in the L^1-norm by periodic $\mathscr{C}^1$-functions, and show that this is obvious for this class; then use a continuity estimate for the maps $g \mapsto a_n,\ n \in \mathbb{Z}$.

2. Let $M : X \to Y$ be linear. Show that the range of M is dense if and only if $\ker M' = \{0\}$.

3. Let X be a Banach space and $T \in \mathscr{L}(X, X)$ with $|T| < 1$.

   (a) Prove that $S_n = \sum_{j=0}^{n} T^j$ is a norm convergent sequence in $\mathscr{L}(X, X)$. Let S denote the limit.

   (b) Prove that for every $y \in X$ we have

   $$(\mathrm{Id} - T)Sy = y \tag{5}$$

   and conclude that $(\mathrm{Id} - T)$ is boundedly invertible. Give a estimate for $|S|$ in terms of $|T|$.

   (c) Show that if $M \in \mathscr{L}(X, X)$ is invertible, then there is an $\epsilon > 0$ so that every $N \in \mathscr{L}(X, X)$ with $|M - N| < \epsilon$ is also invertible. Briefly, invertibility is an open property in the operator topology.

   (d) If $M \in \mathscr{L}(X, X)$, then we define the reolvent set of M to be

   $$\rho(M) = \{\lambda \in \mathbb{C} : (M - \lambda\,\mathrm{Id}) \text{ is invertible }\}. \tag{6}$$

   Prove that $\rho(M)$ is a non-empty, open subset of $\mathbb{C}$.

4. Suppose that X, Y are Banach spaces and $M : X \to Y$ is a surjective, bounded linear map. Show that there is a constant $c > 0$, so that for every $y \in Y$, there exists an $x \in X$ with

   $$Mx = y \text{ and } \|x\| < c\|y\|. \tag{7}$$

   Hint: The induced map $\widetilde{M} : X/N_M \to Y$ is 1-1 and onto.

5. Let $S \subset C^0([0, 1])$ be a subspace, which is closed with respect to the L^2-norm. This means that if $< f_n > \subset S$, and there is a function $f \in L^2[0, 1]$ such that $\| f_n - f \|_{L^2} \to 0$, then f can be represented by a function in S.

   (a) Show that S is also closed as a subspace of C^0.

(b) Show that there is a constant M so that, for $f \in S$, we have

$$\|f\|_\infty < M\|f\|_2. \tag{8}$$

Hint: use the closed graph theorem.

(c) Show that for each $y \in [0,1]$ there is a function $k_y \in L^2([0,1])$ so that for each $f \in S$,

$$f(y) = \int_0^1 f(x)k_y(x)dx. \tag{9}$$

6. (a) Suppose that X and Y are Banach spaces, and $D \subset X$ is a linear subspace, which may not be closed. Suppose that $T : D \to Y$ has a closed graph, (that is $\{(x, Tx) : x \in D\}$ is a closed subspace of $X \times Y$), and T is 1-1 and onto. If D is not closed, then T need not be continuous. Prove, however, that $T^{-1} : Y \to X$ is continuous.

(b) Let X denote continuous functions on $[0,1]$ that vanish at 0; $Y = C^0([0,1])$; and $D \subset X$, those functions with a continuous first derivative. Show that $Tf = \partial_x f$ has a closed graph, and is a 1-1, onto map from D to Y. What is T^{-1}? Give an *elementary* proof that it is bounded as a map from $Y \to X$.