AMCS 609
Problem set 11 due May 5, 2015
Dr. Epstein

Reading: Read Chapters 17, 21, and 22, in Lax, *Functional Analysis*.
Standard problem: The following problems should be done, but do not have to be handed in.

1. For (X, d) a complete metric space, prove that the following definitions of precompact set are equivalent: A set $S \subset X$ is precompact if
   (a) Every sequence of points $< x_n > \subset S$ has a convergent subsequence;
   (b) If for any $\epsilon > 0$, S can be covered by finitely many balls of radius ϵ;
2. Suppose that X is a Banach space. Prove the following statements
   (a) If C_1 and C_2 are precompact, then $C_1 + C_2$ is precompact.
   (b) Let U be another Banach space and $M \in \mathscr{L}(X, U)$. If $C \subset X$ is precompact, then $MC \subset U$ is precompact.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let X be a Banach space, and $K \subset X$ a precompact subset of X. Show that the convex hull of K is also precompact. Hint: Use the covering definition.
2. Let X be a Banach space and $\{P_N : N \in \mathbb{N}\}$ be a sequence of bounded, finite rank operators, which converge strongly to the identity, that is $\lim_{N\to\infty} P_N x = x$, for every $x \in X$. If $C : X \to X$ is a compact operator, then prove that $P_N C$ converges to C in the uniform norm. Show that if X is a Hilbert space, then any compact map $C : X \to X$ is the norm limit of a sequence of finite rank maps. Hint: If H is non-separable, then the sequence $< P_N >$ will not in general converge to the identity.
3. Suppose that X is a Hilbert space and $C : X \to X$ is a compact *self adjoint* operator, that is $\langle Cx, y\rangle = \langle x, Cy\rangle$, for all $x, y \in X$.
   (a) Prove that for all $x \in X$, the function $F(x) = \langle Cx, x\rangle$ is real valued.

(b) Suppose that for some $x,\ F(x) > 0$; show that there is unit vector $x_1 \in X$, so that
$$F(x_1) = \sup\{F(x) :\ x \in X \text{ with } \|x\| = 1\}. \tag{1}$$

(c) Prove that x_1 is an eigenvector of C, that is, there is a real number λ_1 so that $Cx_1 = \lambda_1 x_1$.

(d) If we let $X_1 = \{x \in X :\ \langle x, x_1\rangle = 0\}$, then C maps X_1 to itself, that is $CX_1 \subset X_1$.

4. Let $X = L^2([0,1])$, and define the operator $Mf(x) = xf(x)$. Recall that the spectrum of an operator A is the set

$$\sigma(A) = \{\lambda \in \mathbb{C} :\ (A - \lambda\,\mathrm{Id}) \text{ is not invertible }\}.$$

The complement of the $\sigma(A)$ is called the resolvent set.

(a) Prove that M is a bounded operator, but not a compact operator.

(b) Does there exist a $\lambda \in \mathbb{C}$ and $f \in X$ such that $(M - \lambda\,\mathrm{Id})f = 0$?

(c) What is the spectrum of M? Give a formula for the resolvent operator $R(\lambda) = (M - \lambda\,\mathrm{Id})^{-1}$. Where is it defined?

5. Let $k(s,t)$ be a C^1-function on $[0,1] \times [0,1]$. Define the operator K by

$$Kf(s) = \int_0^1 k(s,t) f(t) dt. \tag{2}$$

Show that $K : C^0([0,1]) \to C^0([0,1])$ and $K : L^2([0,1]) \to L^2([0,1])$ are compact operators.