

AMCS 609

Problem set 11 due April 28, 2011

Dr. Epstein

Reading: Read Chapters 21, 22, 23, 28 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Lax page 244, exercise 10.
2. Lax page 241, exercise 5.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Lax page 240, exercise 4. Do the parts not done in class.
2. Let H be a Hilbert space, and $\{y_n\}$ a set of unit vectors satisfying the hypotheses of Theorem 7 in §22.5.

(a) Show that there is a constant C so that, for all $x \in H$, we have:

$$\frac{1}{C} \|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, y_n \rangle|^2 \leq C \|x\|^2 \quad (1)$$

(b) We can normalize the $\{y_n\}$ so that $\langle x_n, y_n \rangle$ is real, for all n , and define $\{\theta_n\}$ so that $\cos \theta_n = \langle x_n, y_n \rangle$, and $|\theta_n| \leq \frac{\pi}{2}$. Show that

$$\sum_{n=1}^{\infty} |\theta_n|^2 < \infty. \quad (2)$$

(c) For $k \in \mathbb{N}$ set $n_k = k + \epsilon_k$; find a non-trivial condition on $\{\epsilon_k\}$, so that the set of functions $\{\sin n_k x : k \in \mathbb{N}\}$ defines a basis for $L^2([0, \pi])$.

3. Define

$$Kf(x) = \int_0^{\pi} \log|x-y|f(y)dy. \quad (3)$$

Show that $K : L^2[0, \pi] \rightarrow L^2[0, \pi]$ and $K : C^0[0, \pi] \rightarrow C^0[0, \pi]$ are both compact operators.

4. Suppose that we define the projection operator on $L^2(S^1)$, in terms of the Fourier representation by

$$P \left(\sum_{j=-\infty}^{\infty} a_j e^{ij\theta} \right) = \sum_{j=0}^{\infty} a_j e^{ij\theta}. \quad (4)$$

- (a) Show that P is self adjoint.
 (b) Let $g \in C^0(S^1; \mathbb{C})$, define the multiplication operator $M_g f = gf$. Show that the composition satisfies:

$$\|M_g f\|_{L^2} \leq \|g\|_{L^\infty} \|f\|_{L^2}. \quad (5)$$

What is the adjoint of PM_g ?

- (c) Let g be a trigonometric polynomial:

$$g = \sum_{j=-N}^N b_j e^{ij\theta} \quad (6)$$

Show that the commutator, $[P, M_g] = PM_g - M_g P$, is a finite rank operator.

- (d) Show that if $g \in C^0(S^1; \mathbb{C})$, then the commutator $[P, M_g]$ is a compact operator. Hint: Approximate g .
 (e) If we let $H^2(S^1)$ denote the range of P , then we see that $PM_g : H^2(S^1) \rightarrow H^2(S^1)$. We sometimes write this operator as $PM_g P$. If $g_1, g_2 \in C^0(S^1; \mathbb{C})$, then prove that

$$PM_{g_1} P PM_{g_2} P - PM_{g_1} M_{g_2} P \quad (7)$$

is a compact operator.

- (f) Show that if g is a non-vanishing continuous function on S^1 , then there is an operator $A : H^2(S^1) \rightarrow H^2(S^1)$ so that

$$PM_g P A - \text{Id} \text{ and } A PM_g P - \text{Id} \quad (8)$$

are compact operators.

- (g) Find non-vanishing, continuous functions g_1, g_2 on S^1 so that $\ker PM_{g_1} P$ is non-trivial, and the range of $PM_{g_2} P$ is not all of $H^2(S^1)$.