

AMCS 609

Problem set 2 due February 8, 2011

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Reading: Read Chapters 1, 2, and 3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Suppose that $K, L \subset X$, a real vector space, are convex sets. Prove that $K + L$ is also convex.
2. Let X, Y be real vector spaces and $M : X \rightarrow Y$ a linear map. Prove that if $K \subset X$ is convex, then $M(K)$ is convex, and if $L \subset Y$ is convex, then $M^{-1}(L)$ is convex.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. A linear function from a real vector space X to \mathbb{R} is just a linear map $\ell : X \rightarrow \mathbb{R}$. Show that a linear function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous with respect to the topology defined by any norm on \mathbb{R}^n .
2. Let X be a finite dimensional vector space, and $Y \subset X$ a proper subspace. Let $\{y_1, \dots, y_k\}$ be a basis for Y . If $\dim X = n$, then show that there are vectors $\{x_1, \dots, x_{n-k}\}$ so that $\{y_1, \dots, y_k, x_1, \dots, x_{n-k}\}$ is a basis for X . Conclude that

$$\dim X = \dim Y + \dim(X/Y). \quad (1)$$

3. Suppose that X is a finite dimensional real vector space.
 - (a) Show that the set, X' , of linear functions on X , with its natural vector space structure, has the same dimension as X . If $Y \subset X$ is a subspace, then the $\dim(X/Y)$ is called the *codimension* of Y , and

$$Y^\perp = \{\ell \in X' : \ell(y) = 0 \text{ for all } y \in Y\}. \quad (2)$$

- (b) Show that Y^\perp is a subspace of X' and $\dim(X/Y) = \dim Y^\perp$.

- (c) Let $d \in \mathbb{N}$, and \mathcal{P}_d denote polynomials with real coefficients of order at most d . Show that the functionals

$$\ell_j(p) = \partial_x^j p(0) \text{ for } j = 0, \dots, d \quad (3)$$

are a basis for \mathcal{P}_d . For $0 \leq d' < d$ use this basis to describe \mathcal{P}_d^\perp .

4. Let X be a finite dimensional vector space over \mathbb{C} , and let $X_{\mathbb{R}}$ denote the vector space X , but with the scalar multiplication restricted to the real numbers. Prove that $\dim_{\mathbb{R}} X_{\mathbb{R}} = 2 \dim_{\mathbb{C}} X$. Show that $z \mapsto \bar{z}$ is a linear map from $\mathbb{C}_{\mathbb{R}} \rightarrow \mathbb{C}_{\mathbb{R}}$, but not from $\mathbb{C} \rightarrow \mathbb{C}$.
5. Suppose that $K \subset \mathbb{R}^2$ is a convex set. A point x lies on the boundary of K , bK , if, for any $\epsilon > 0$, $B_\epsilon(x) \cap K \neq \emptyset$, and $B_\epsilon(x) \cap K^c \neq \emptyset$. Show that if $x \in bK$, then there is a linear function $\ell_x : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that

$$\ell_x(x) \geq \ell_x(y) \text{ for all } y \in K \setminus \{x\}. \quad (4)$$

When does the strict inequality hold for all $y \in K \setminus \{x\}$? The set $\{y : \ell_x(y) = \ell_x(x)\}$ is called a supporting line. Is the supporting line always unique?

6. Let $\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a linear function. A set of the form

$$H_{\ell,c} = \{x \in \mathbb{R}^2 : \ell(x) > c\} \quad (5)$$

is called an open half space. If $K \subset \mathbb{R}^2$ is a closed convex set, then show that

$$K = \bigcap_{H_{\ell,c} \supset K} H_{\ell,c}. \quad (6)$$

That is, K is the intersection of all the open half spaces that contain it. Prove that a closed unbounded, proper convex subset of \mathbb{R}^2 satisfies exactly one of the following criteria:

- (a) K is a closed half space.
 (b) K is the region between two parallel lines.
 (c) K lies in a proper cone (the intersection of two half-spaces with non-parallel boundaries).

7. Let $X = \mathbb{R}^2$ and $Y = \{(x, 0) : x \in \mathbb{R}\}$, be a subspace. Suppose that we define a linear function ℓ on Y by setting $\ell((1, 0)) = 1$. For $1 \leq p < \infty$, define the norms

$$\|(x, y)\|_p = (x^p + y^p)^{\frac{1}{p}}, \quad (7)$$

and

$$\|(x, y)\|_\infty = \max\{|x|, |y|\}. \quad (8)$$

This linear function on Y satisfies

$$|\ell((x, 0))| \leq \|(x, 0)\|_p, \quad (9)$$

for all $1 \leq p \leq \infty$. We can linearly extend ℓ to all of \mathbb{R}^2 by setting

$$\ell((0, 1)) = \beta. \quad (10)$$

Denote this extension by ℓ_β . For each $1 \leq p \leq \infty$, find the values of β so that

$$|\ell_\beta((x, y))| \leq \|(x, y)\|_p, \text{ for all } (x, y) \in \mathbb{R}^2. \quad (11)$$

We can define another family of norms, for $0 < a < \infty$, by setting

$$N_a(x, y) = \sqrt{x^2 + a^2 y^2}. \quad (12)$$

For each $0 < a < \infty$, find the values of β so that

$$|\ell_\beta((x, y))| \leq N_a(x, y), \text{ for all } (x, y) \in \mathbb{R}^2. \quad (13)$$