# AMCS/MATH 609
## Problem set 2 due February 3, 2015
## Dr. Epstein

**Reading:** There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi. **Standard problems:** The solutions to the following problems do not need to be handed in.

1. Let $\varphi$ be an integrable function defined in $\mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d} \varphi(x)dx = 1. \tag{1}$$

   We let $K_\delta(x) = \delta^{-d}\varphi\left(\frac{x}{\delta}\right)$.

   (a) Prove that $K_\delta$ is a family of good kernels.

   (b) Show that if $\varphi$ is bounded and supported in a bounded set, then $K_\delta$ defines an approximation to the identity.

2. Let $E \subset [0, 1]$ be a measurable subset for which there exists an $\alpha > 0$, so that $m(E \cap I) \geq \alpha m(I)$ for every interval $I \subset [0, 1]$. Show that $m(E) = 1$.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Show that if $K_\delta$ is a family of good kernels, then for any $f \in L^1(\mathbb{R}^d)$

$$\lim_{\delta\to 0^+} \|K_\delta * f - f\|_{L^1(\mathbb{R}^d)} = 0. \tag{2}$$

2. Suppose that $\{K_\delta(x) : 0 < \delta\}$ is a family of kernels defined in $\mathbb{R}^d$ that satisfies:

   (a) $|K_\delta(x)| \leq A\delta^{-d}$, for all $0 < \delta$.

   (b) $|K_\delta(x)| \leq A\delta/|x|^{d+1}$, for all $0 < \delta$.

   (c) $\int_{\mathbb{R}^d} K_\delta(x)dx = 0$, for all $0 < \delta$.

Show that if $f \in L^1(\mathbb{R}^d)$, then

$$\lim_{\delta \to 0^+} K_\delta * f(x) = 0 \text{ for a.e. } x. \tag{3}$$

3. Let

$$f(x) = \sum_{j=1} \alpha_j \chi_{E_j}(x) \tag{4}$$

be a non-negative, simple function in canonical form, and define its distribution function:

$$\lambda(s) = |\{x : f(x) > s\}|. \tag{5}$$

For $p > 0$ show that

$$\int_{\mathbb{R}^d} |f(x)|^p = p \int_0^\infty s^{p-1} \lambda(s) ds. \tag{6}$$

Give a definition for $d\lambda(s)$ as a measure on the Borel sets of $[0, \infty]$ so that

$$p \int_0^\infty s^{p-1} \lambda(s) ds = -\int_0^\infty s^p d\lambda(s). \tag{7}$$

Prove that these two formulæ hold for any function for which

$$\int_{\mathbb{R}^d} |f(x)|^p dx < \infty. \tag{8}$$

4. Let $F$ be a closed subset in $\mathbb{R}$, and let

$$\delta(x) = \inf\{|x - y| : y \in F\}. \tag{9}$$

Clearly $\delta(x + y) \leq |y|$ whenever $x \in F$. Prove the more refined estimate

$$\delta(x + y) = o(|y|), \text{ that is } \lim_{y \to 0} \frac{\delta(x + y)}{|y|} = 0, \text{ for a.e. } x \in F. \tag{10}$$

Hint: Assume that $x$ is a point of density of $F$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. We define the completion of this space as follows: Let $\overline{\mathcal{M}}$ be of collection of sets of the form $E \cup Z$, where $E \in \mathcal{M}$ and $Z \subset F$, where $F \in \mathcal{M}$ with $\mu(F) = 0$. We also define $\overline{\mu}(E \cup Z) = \mu(E)$.

   (a) Show that $\overline{\mathcal{M}}$ is the smallest $\sigma$-algebra containing $\mathcal{M}$ and all subsets of elements of $\mathcal{M}$, which have measure zero.

   (b) Show that $\overline{\mu}$ is a measure, and this measure is complete.