

AMCS 609

Problem set 3 due February 15, 2011

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Reading: Read Chapters 3.2-3.3 (especially the proof of Theorem 8), 4.2, 5.1-5.2, 6.1-6.3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Prove Theorem 4 in §3.2 of Lax.
2. Prove Corollary 5' of §3.2 of Lax.
3. Suppose that (X, d) is a metric space. Show that if $\lim_{n \rightarrow \infty} x_n = x^*$, then, for any $x \in X$, we also have that

$$\lim_{n \rightarrow \infty} d(x, x_n) = d(x, x^*). \quad (1)$$

4. For $1 \leq p \leq \infty$, prove that the normed vector space ℓ_p is a Banach space.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that in any real normed linear space $(X, \|\cdot\|)$, the open and closed unit balls

$$B_1 = \{x \in X : \|x\| < 1\}, \quad \overline{B}_1 = \{x \in X : \|x\| \leq 1\} \quad (2)$$

are convex and have non-empty interior. The unit ball is *strictly convex*, if, whenever $\|x\| = \|y\| = 1$ and $x \neq y$, then

$$\left\| \frac{x+y}{2} \right\| < 1. \quad (3)$$

Show that the unit ball in ℓ_2 is strictly convex, but the unit ball in ℓ_1 is not.

2. A sequence $\langle c_j \rangle$ is Cesaro summable if

$$\lim_{n \rightarrow \infty} \frac{c_1 + \cdots + c_n}{n} \text{ exists.} \quad (4)$$

Show that a Banach limit LIM can be defined so that if $\langle c_j \rangle$ is Cesaro summable then

$$\text{LIM} c_j = \lim_{n \rightarrow \infty} \frac{c_1 + \cdots + c_n}{n}. \quad (5)$$

3. Suppose that X is a Banach space and $Y \subset X$ is a closed subspace. Show that the quotient space X/Y , with the quotient norm

$$\|[x]\|_{X/Y} = \inf_{x \in [x]} \|x\|_X, \quad (6)$$

is complete.

4. Prove that every finite dimensional subspace of a normed vector space is closed. Hint: Use the fact that all norms on a finite dimensional vector space are equivalent to show that every finite dimensional subspace is complete.
5. Let $Y \subset \ell_\infty$ be the subspace of sequences that are eventually zero (only finitely many terms non-zero). Find the closure of Y with respect to the ℓ_∞ -norm.
6. Prove that ℓ_1 has a countable dense subset, but ℓ_∞ does not.
7. Let $\mathcal{H}^2(D_1)$ denote the closure of bounded holomorphic functions on the unit disk with respect to the L^2 -norm

$$\|f\|_2^2 = \int_{D_1} |f(x, y)|^2 dx dy = \lim_{r \rightarrow 1^+} \iint_{D_r} |f(x, y)|^2 dx dy < \infty. \quad (7)$$

$L^2(D_1)$ is defined as the closure of $C^0(\overline{D}_1)$ with respect to the L^2 -norm.

- (a) Show that $f \in \mathcal{H}^2(D_1)$ is holomorphic in $\text{int } D_1$.
- (b) Show that if f is a square integrable function in D_1 , which is holomorphic in the interior of D_1 , then $f \in \mathcal{H}^2(D_1)$.
- (c) Prove that for any $k \in \mathbb{N}$, there is a bounded linear functional ℓ_k defined on $L^2(D_1)$, so that if $f \in \mathcal{H}^2(D_1)$, then

$$\ell_k(f) = \partial_z^k f(0). \quad (8)$$