

AMCS 609

Problem set 3 due February 10, 2009

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Reading: Read Chapters 3.2-3.3 (especially the proof of Theorem 8), 4.2, 5.1-5.2, 6.1-6.3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Prove Theorem 4 in §3.2 of Lax.
2. Prove Corollary 5' of §3.2 of Lax.
3. Prove that in any real normed linear space $(X, \|\cdot\|)$, the open and closed unit balls

$$B_1 = \{x \in X : \|x\| < 1\}, \quad \overline{B}_1 = \{x \in X : \|x\| \leq 1\} \quad (1)$$

are convex and have non-empty interior. The unit ball is *strictly convex*, if, whenever $\|x\| = \|y\| = 1$ and $x \neq y$, then

$$\left\| \frac{x+y}{2} \right\| < 1. \quad (2)$$

Show that the unit ball in ℓ_2 is strictly convex, but the unit ball in ℓ_1 is not.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Exercise 2 in §4.2. A sequence $\langle c_j \rangle$ is Cesaro summable if

$$\lim_{n \rightarrow \infty} \frac{c_1 + \cdots + c_n}{n} \text{ exists.} \quad (3)$$

2. Exercise 3 in §5.1.
3. Exercise 4 in §5.2.
4. Exercise 1, 2, and 3 in §6.1.
5. For $1 \leq p \leq \infty$, prove that the normed vector space ℓ_p is a Banach space.

6. Let $Y \subset \ell_\infty$ be the subspace of sequences that are eventually zero (only finitely many terms non-zero). Find the closure of Y with respect to the ℓ_∞ -norm.
7. Let $(H, (\cdot, \cdot))$ be a Hilbert space and $Y \subset H$ a closed subspace. As proved in class, every vector $x \in H$, has a unique representation $x = y + y^\perp$, with $y \in Y$ and $y^\perp \in Y^\perp$. This defines a map $P : H \rightarrow Y$, with $Px = y$. Prove that P is a linear map, and that $P^2 = P$. Show that the quotient norm induced on H/Y^\perp is given by

$$\|[x]\|_{H/Y^\perp} = \|P\tilde{x}\|_H, \text{ for any } \tilde{x} \in [x]. \quad (4)$$

8. Prove that ℓ_1 has a countable dense subset, but ℓ_∞ does not.