AMCS/MATH 609
Problem set 4 due February 24, 2015
Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi. **Standard problems:** The solutions to the following problems do not need to be handed in.

1. Suppose that $L : \mathbb{R}^d \to \mathbb{R}^d$ is a linear transformation. Show that if E is a Lebesgue measurable set then so is $L(E)$. Hint: Show that L maps sets of measure zero to sets of measure zero. Prove that

$$m(L(E)) = |\det(L)|m(E). \tag{1}$$

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that ν, ν_1, ν_2 are signed measures on $(X, \mathcal{M})$ and μ is a positive measure. Prove the following assertions:

   (a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $(\nu_1 + \nu_2) \perp \mu$.

   (b) If $\nu_1 << \mu$ and $\nu_2 << \mu$, then $(\nu_1 + \nu_2) << \mu$.

   (c) If $\nu_1 \perp \nu_2$ implies that $|\nu_1| \perp |\nu_2|$.

   (d) $\nu << |\nu|$.

   (e) If $\nu \perp \mu$ and $\nu << \mu$, then $\nu = 0$.

2. If $\nu << \mu$, with μ a positive, σ-finite measure, then we let $\frac{d\nu}{d\mu}$ denote the Radon-Nikodym derivative, so that

$$\int_E d\nu = \int_E \left[\frac{d\nu}{d\mu}\right] d\mu. \tag{2}$$

   (a) If $\nu << \mu$ and f is a non-negative measureable function, then

$$\int_X f(x) d\nu(x) = \int_X f(x) \left[\frac{d\nu}{d\mu}\right](x) d\mu(x). \tag{3}$$

(b) If $\nu_1 << \mu$ and $\nu_2 << \mu$, then

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu} \tag{4}$$

(c) If $\lambda << \nu << \mu$, with ν and μ positive measures, then

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}. \tag{5}$$

(d) If $\nu << \mu$, and $\mu << \nu$, with both measures positive, then

$$\frac{d\nu}{d\mu} = \left[\frac{d\mu}{d\nu}\right]^{-1} \tag{6}$$

3. In this problem we give an example to show that the σ-finiteness of μ cannot be omitted from the hypotheses of the Radon-Nikodym theorem. Let $X = [0, 1]$ and $\mathcal{M}$ be the class of Lebesgue measurable subsets of $[0, 1]$. Let ν be Lebesgue measure restricted to X and μ be the counting measure on subsets of X. Clearly $\nu << \mu$, but show that there is no measurable function f such that

$$\nu(E) = \int_E f(x) d\mu(x). \tag{7}$$

4. Suppose that μ_1, ν_1 are σ-finite measures on $(X_1, \mathcal{M}_1)$ and μ_2, ν_2 are σ-finite measures on $(X_2, \mathcal{M}_2)$, with μ_1 and μ_2 positive measures. Show that if $\nu_1 << \mu_1$ and $\nu_2 << \mu_2$, then $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \cdot \frac{d\nu_2}{d\mu_2}(x_2). \tag{8}$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing, continuously differentiable function. Show that f maps Borel measureable sets to Borel measurable sets. Define a Borel measure by setting

$$\mu(E) = m(f(E)), \tag{9}$$

where m is Lebesgue measure. Show that $\mu << m$, and compute the Radon-Nikodym derivative $\frac{d\mu}{dm}$.

6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, also σ-finite. We let $\nu = \mu \restriction_{\mathcal{N}}$.

(a) Show that for any $f \in L^1(X; d\mu)$ there is a function $g \in L^1(X; d\nu)$ (which is therefore $\mathcal{N}$-measurable) so that for any set $E \in \mathcal{N}$, we have

$$\int_E f(x)d\mu(x) = \int_E g(x)d\nu(x). \tag{10}$$

The point here is that g is measurable with respect to $\mathcal{N}$, while in general f is not. Show that g is unique modulo sets of ν measure zero.

(b) Suppose that $\mathcal{M}$ is the Lebesgue measurable subsets of $\mathbb{R}$ and $\mathcal{N}$ is the σ-algebra generated by the sets $\{(n, n+1] : n \in \mathbb{Z}\}$. Give a formula for g in terms of f.