

AMCS 609

Problem set 4 due February 22, 2011

Dr. Epstein

Reading: Read Chapters 6.3-4, 7.1-2, and B1, B2 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Exercise 3 on page 54 of Lax.
2. Exercise 9 on page 61 of Lax.
3. Exercise 10 on page 62 of Lax.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $(H, (\cdot, \cdot))$ be a Hilbert space and $Y \subset X$ a closed subspace. As proved in class, every vector $x \in H$, has a unique representation $x = y + y^\perp$, with $y \in Y$ and $y^\perp \in Y^\perp$. This defines a map $P : H \rightarrow Y$, with $Px = y$. Prove that P is a linear map, and that $P^2 = P$. Show that the quotient norm induced on H/Y^\perp is given by

$$\|[x]\|_{H/Y^\perp} = \|P\tilde{x}\|_H, \text{ for any } \tilde{x} \in [x]. \quad (1)$$

A linear map satisfying $P^2 = P$ is called a *projection*; can you explain why?

2. We can define an inner product on $\mathcal{C}_0^\infty(0, 1)$ by setting

$$\langle f, g \rangle_1 = \int_0^1 f'(x)g'(x)dx. \quad (2)$$

We let $\|f\|_1$ denote the norm defined by this inner product.

- (a) Show that

$$\int_0^1 |f(x)|^2 dx \leq \int_0^1 |f'(x)|^2 dx. \quad (3)$$

Explain why this shows that the norm is non-degenerate.

- (b) We let H_1 denote the completion of $\mathcal{C}_0^\infty(0, 1)$ with respect to $\|\cdot\|_1$. Show that for $f \in H_1$, and $x, y \in [0, 1]$ we have the following estimate

$$|f(x) - f(y)| \leq \|f\|_1 \sqrt{|x - y|}. \quad (4)$$

A limiting argument is required to prove this; show that this estimate implies that the elements of H_1 are represented by continuous functions. Show that, such representatives of $f \in H_1$ satisfy $f(0) = f(1) = 0$.

- (c) Show that for each $x \in (0, 1)$ the linear functional

$$\ell_x(f) = f(x), \quad (5)$$

is bounded on H_1 .

- (d) For each $x \in (0, 1)$ find the unique element $g_x \in H_1$, so that

$$\ell_x(f) = \langle f, g_x \rangle_1. \quad (6)$$

Be careful that the element $g_x \in H_1$, in particular that $g_x(0) = g_x(1) = 0$.

- (e) Describe the orthogonal complement of the subspace

$$\ker \ell_x = \{f \in H_1 : \ell_x(f) = 0\}.$$

3. Let $\mathcal{H}^2(D_1)$ be the square integrable holomorphic functions in D_1 , with

$$\|f\|_2^2 = \int_{D_1} |f(x, y)|^2 dx dy. \quad (7)$$

- (a) Show that there are positive constants $\{c_n\}$ so that $\{c_n z^n : n = 0, 1, \dots\}$ is an orthonormal basis for $\mathcal{H}^2(D_1)$. You need to prove that the basis is complete!
- (b) For each $w \in D_1$, we can define a linear functional

$$\ell_w(f) = f(w). \quad (8)$$

In the previous problem set we showed that ℓ_w is a bounded linear functional. The Riesz Representation Theorem shows that there is a unique $g_w \in \mathcal{H}^2(D_1)$ so that, for all $f \in \mathcal{H}^2(D_1)$, we have

$$f(w) = \int_{D_1} f(z) \overline{g_w(z)} dx dy. \quad (9)$$

Prove that g_w solves the following variational problem: Find a function $g \in \mathcal{H}^2(D_1) \setminus \{0\}$ such that

$$F(g) = \frac{|g(w)|}{\|g\|_2} \quad (10)$$

is maximized. Suppose that $M_w = \sup_{g \neq 0} F(g)$, and f is a function such that $F(f) = M_w$. Prove that $f = \lambda g_w$, for a $\lambda \in \mathbb{C}$.

- (c) Show that for each $N \in \mathbb{N}$ the projection onto $\text{span}(1, z, z^2, \dots, z^N)$ is given by

$$[P_N f](w) = \int_{D_1} k_N(z, w) f(z) dx dy, \quad (11)$$

where

$$k_N(z, w) = \sum_{n=0}^N c_n^2 \bar{z}^n w^n \quad (12)$$

- (d) Use the observation in the previous part to show that

$$g_w(z) = \frac{1}{\pi(1 - z\bar{w})^2}. \quad (13)$$

Compute M_w . Hint: This does not require any complicated computations.