

AMCS 609  
Problem set 4 due February 17, 2009  
Dr. Epstein

**Reading:** Read Chapters 6.3-4, 7.1-2, and B1, B2 in Lax, *Functional Analysis*.

**Standard problem:** The following problems should be done, but do not have to be handed in.

1. Exercise 3 on page 54.
2. Exercise 9 on page 61.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. We can define an inner product on  $\mathcal{C}_0^\infty(0, 1)$  by setting

$$\langle f, g \rangle_1 = \int_0^\infty f'(x)g'(x)dx. \quad (1)$$

We let  $\|f\|_1$  denote the norm defined by this inner product.

- (a) Show that

$$\int_0^1 |f(x)|^2 dx \leq 4 \int_0^1 |f'(x)|^2 dx. \quad (2)$$

Explain why this shows that the norm is non-degenerate.

- (b) We let  $H_1$  denote the completion of  $\mathcal{C}_0^\infty(0, 1)$  with respect to  $\|\cdot\|_1$ . Show that for  $f \in H_1$ , and  $x, y \in [0, 1]$  we have the following estimate

$$|f(x) - f(y)| \leq \|f\|_1 \sqrt{|x - y|}. \quad (3)$$

A limiting argument is required to prove this; show that this estimate implies that the elements of  $H_1$  are represented by continuous functions.

- (c) Show that for each  $x \in (0, 1)$  the linear functional

$$\ell_x(f) = f(x), \quad (4)$$

is bounded on  $H_1$ .

(d) For each  $x \in (0, 1)$  find the unique element  $g_x \in H_1$ , so that

$$\ell_x(f) = \langle f, g_x \rangle_1. \quad (5)$$

2. Let  $\mathcal{H}^2(D_1)$  be the square integrable holomorphic functions in  $D_1$ , with

$$\|f\|_2^2 = \int_{D_1} |f(x, y)|^2 dx dy. \quad (6)$$

(a) Show that there are positive constants  $\{c_n\}$  so that  $\{c_n z^n : n = 0, 1, \dots\}$  is an orthonormal basis for  $\mathcal{H}^2(D_1)$ . You need to prove that the basis is complete!

(b) For each  $w \in D_1$ , we can define a linear functional

$$\ell_w(f) = f(w). \quad (7)$$

Prove that  $\ell_w$  is a bounded linear functional.

(c) The Riesz Representation Theorem shows that there is a unique  $g_w \in \mathcal{H}^2(D_1)$  so that, for all  $f \in \mathcal{H}^2(D_1)$ , we have

$$f(w) = \int_{D_1} f(z) \overline{g_w(z)} dx dy. \quad (8)$$

Prove that  $g_w$  solves the following variational problem: Find a function  $g \in \mathcal{H}^2(D_1) \setminus \{0\}$  such that

$$F(g) = \frac{|g(w)|}{\|g\|_2} \quad (9)$$

is maximized. Suppose that  $M_w = \sup_{g \neq 0} F(g)$ , and  $f$  is a function such that  $F(f) = M_w$ . Prove that  $f = \lambda g_w$ , for a  $\lambda \in \mathbb{C}$ .

(d) Show that for each  $N \in \mathbb{N}$  the projection onto  $\text{span}(1, z, z^2, \dots, z^N)$  is given by

$$P_N f(z) = \int_{D_1} k_N(z, w) f(z) dx dy, \quad (10)$$

where

$$k_n(z, w) = \sum_{n=0}^N c_n^2 \bar{z}^n w^n \quad (11)$$

(e) Use the observation in the previous part to show that

$$g_w(z) = \frac{1}{\pi(1 - z\bar{w})^2}. \quad (12)$$

Compute  $M_w$ . Hint: This does not require any complicated computations.

3. For  $f \in \mathcal{C}_0^\infty(\mathbb{R})$ , show that  $|f|$  has a weak derivative, which can be represented by a function  $g(x)$  that satisfies

$$|g(x)| \leq |\partial_x f(x)|. \quad (13)$$

The statement that the weak derivative is *represented by*  $g$  means that for all  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$  we have

$$\int_{\mathbb{R}} |f(x)| \partial_x \varphi(x) dx = - \int_{\mathbb{R}} g(x) \varphi(x) dx. \quad (14)$$

Hint: Consider  $\sqrt{f^2(x) + \epsilon^2}$ . Compute the weak derivative of  $|x|$ .

4. Exercise 10 on page 62 of Lax.