AMCS/MATH 609
Problem set 5 due March 3, 2015
Dr. Epstein

Reading: Read Chapters 1, 2, and 3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Let X be a finite dimensional vector space, and $Y \subset X$ a proper subspace. Let $\{y_1, \dots, y_k\}$ be a basis for Y. If $\dim X = n$, then show that there are vectors $\{x_1, \dots, x_{n-k}\}$ so that $\{y_1, \dots, y_k, x_1, \dots, x_{n-k}\}$ is a basis for X. Conclude that
$$\dim X = \dim Y + \dim(X/Y). \tag{1}$$

2. A linear function from a real vector space X to $\mathbb{R}$ is just a linear map $\ell : X \to \mathbb{R}$. Show that a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to the topology defined by any norm on $\mathbb{R}^n$.

3. Suppose that $K, L \subset X$, a real vector space, are convex sets. Prove that $K + L$ is also convex.

4. Let X, Y be real vector spaces and $M : X \to Y$ a linear map. Prove that if $K \subset X$ is convex, then $M(K)$ is convex, and if $L \subset Y$ is convex, then $M^{-1}(L)$ is convex.

5. Let X be a finite dimensional vector space over $\mathbb{C}$, and let $X_{\mathbb{R}}$ denote the vector space X, but with the scalar multiplication restricted to the real numbers. Prove that $\dim_{\mathbb{R}} X_{\mathbb{R}} = 2\dim_{\mathbb{C}} X$. Show that $z \mapsto \bar{z}$ is a linear map from $\mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$, but not from $\mathbb{C} \to \mathbb{C}$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that X is a finite dimensional real vector space.

   (a) Show that the set, X', of linear functions on X, with its natural vector space structure, has the same dimension as X. If $Y \subset X$ is a subspace, then the $\dim(X/Y)$ is called the *codimension* of Y, and
$$Y^{\perp} = \{\ell \in X' : \ell(y) = 0 \text{ for all } y \in Y\}. \tag{2}$$

(b) Show that $Y^{\perp}$ is a subspace of X' and $\dim(X/Y) = \dim Y^{\perp}$.

(c) Let $d \in \mathbb{N}$, and $\mathcal{P}_d$ denote polynomials with real coefficients of order at most d. Show that the functionals

$$\ell_j(p) = \partial_x^j p(0) \text{ for } j = 0, \dots, d \tag{3}$$

are a basis for $\mathcal{P}'_d$. For $0 \leq d' < d$ use this basis to describe $\mathcal{P}_{d'}^{\perp}$.

2. Suppose that $K \subset \mathbb{R}^2$ is a convex set. A point x lies on the boundary of K, bK, if, for any $\epsilon > 0$, $B_\epsilon(x) \cap K \neq \emptyset$, and $B_\epsilon(x) \cap K^c \neq \emptyset$. Show that if $x \in bK$, then there is a linear function $\ell_x : \mathbb{R}^2 \to \mathbb{R}$ so that

$$\ell_x(x) \geq \ell_x(y) \text{ for all } y \in K \setminus \{x\}. \tag{4}$$

When does the strict inequality hold for all $y \in K \setminus \{x\}$? The set $\{y : \ell_x(y) = \ell_x(x)\}$ is called a supporting line. Is the supporting line always unique?

3. Let $\ell : \mathbb{R}^2 \to \mathbb{R}$ be a linear function. A set of the form

$$H_{\ell,c} = \{x \in \mathbb{R}^2 : \ \ell(x) > c\} \tag{5}$$

is called an open half space. If $K \subset \mathbb{R}^2$ is a closed convex set, then show that

$$K = \bigcap_{H_{\ell,c} \supset K} H_{\ell,c}. \tag{6}$$

That is, K is the intersection of all the open half spaces that contain it. Prove that a closed unbounded, proper convex subset of $\mathbb{R}^2$ satisfies exactly one of the following criteria:

(a) K is a closed half space.

(b) K is the region between two parallel lines.

(c) K lies in a proper cone (the intersection of two half-spaces with non-parallel boundaries).

4. Let $X = \mathbb{R}^2$ and $Y = \{(x, 0) : \ x \in \mathbb{R}\}$, be a subspace. Suppose that we define a linear function ℓ on Y by setting $\ell((1, 0)) = 1$. For $1 \leq p < \infty$, define the norms

$$\|(x, y)\|_p = (|x|^p + |y|^p)^{\frac{1}{p}}, \tag{7}$$

and

$$\|(x, y)\|_\infty = \max\{|x|, |y|\}. \tag{8}$$

This linear function on Y satisfies

$$|\ell((x, 0))| \le \|(x, 0)\|_p, \tag{9}$$

for all $1 \le p \le \infty$. We can linearly extend ℓ to all of $\mathbb{R}^2$ by setting

$$\ell((0, 1)) = \beta. \tag{10}$$

Denote this extension by ℓ_β. For each $1 \le p \le \infty$, find the values of β so that

$$|\ell_\beta((x, y))| \le \|(x, y)\|_p, \text{ for all } (x, y) \in \mathbb{R}^2. \tag{11}$$

We can define another family of norms, for $0 < a < \infty$, by setting

$$N_a(x, y) = \sqrt{x^2 + a^2 y^2}. \tag{12}$$

For each $0 < a < \infty$, find the values of β so that

$$|\ell_\beta((x, y))| \le N_a(x, y), \text{ for all } (x, y) \in \mathbb{R}^2. \tag{13}$$

5. Show for $0 < q < 1$, the function $d_q : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$$d_q(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{n} |x_i - y_i|^q \tag{14}$$

defines a metric on $\mathbb{R}^n$. How about $d_q(\boldsymbol{x}, \boldsymbol{y})^{\frac{1}{q}}$? What is

$$\lim_{q \to 0^+} d_q(\boldsymbol{x}, \boldsymbol{y})? \tag{15}$$