

AMCS 609

Problem set 5 due March 1, 2011

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Reading: Read Chapters 8.1-3, and 9.1 in Lax, *Functional Analysis*. **Standard problem:** The following problems should be done, but do not have to be handed in.

1. Let $\{V_1, \dots, V_N\}$ be linearly independent vectors in a complex inner product space $(X, \langle \cdot, \cdot \rangle)$. Show that the matrix

$$G_{ij} = \langle V_i, V_j \rangle, \quad (1)$$

is Hermitian symmetric and positive definite, that is $G_{ij} = \overline{G_{ji}}$ and there is a positive constant C so that for $(v_1, \dots, v_n) \in \mathbb{C}^N$, we have

$$\sum_{i,j=1}^N G_{ij} v_i \bar{v}_j \geq C \sum_{j=1}^N |v_j|^2. \quad (2)$$

Show that G_{ij} is invertible. Prove that for $(v_1, \dots, v_n), (w_1, \dots, w_N) \in \mathbb{C}^N$, we have

$$\left| \sum_{i,j=1}^N G_{ij} v_i \bar{w}_j \right| \leq \sqrt{\sum_{i,j=1}^N G_{ij} v_i \bar{v}_j} \sqrt{\sum_{i,j=1}^N G_{ij} w_i \bar{w}_j} \quad (3)$$

2. Exercises 1 and 2 on page 76 of Lax.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. For $f \in \mathcal{C}_0^\infty(\mathbb{R})$, show that $|f|$ has a weak derivative, which can be represented by a function $g(x)$ that satisfies

$$|g(x)| \leq |\partial_x f(x)|. \quad (4)$$

The statement that the weak derivative is *represented by* g means that for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} |f(x)| \partial_x \varphi(x) dx = - \int_{\mathbb{R}} g(x) \varphi(x) dx. \quad (5)$$

Hint: Consider $\sqrt{f^2(x) + \epsilon^2}$. Compute the weak derivative of $|x|$.

2. Suppose that $f \in C^0([0, 1])$ and for every function $\varphi \in \mathcal{C}_0^\infty((0, 1))$, we have

$$\int_0^1 f(x)\varphi(x)dx = 0, \quad (6)$$

prove that $f = 0$ in $C^0([0, 1])$. Now show that if $f \in L^2([0, 1])$ and this condition holds for all $\varphi \in \mathcal{C}_0^\infty((0, 1))$, then $f = 0$ in $L^2([0, 1])$. Remember that $L^2([0, 1])$ is the closure of $C^0([0, 1])$ with respect to the L^2 -norm. The proofs are completely different in the two cases!

3. A function u , which belongs to $L^2([-R, R])$ for all $R > 0$, is weakly constant if

$$\int u(x)\partial_x\varphi(x) = 0 \quad (7)$$

for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Show that a weakly constant function is smooth and constant, or more accurately: has a smooth representative, which is constant. Hint: Show that if $u(x)$ is weakly constant then so is $au(x - y)$ for all $a, y \in \mathbb{R}$.

4. Suppose that we define a weak solution of the wave equation,

$$\partial_x^2 u(x, t) - \partial_t^2 u(x, t) = 0, \quad (8)$$

to be a function that is square integrable in $[-R, R] \times [-R, R]$ for any R , and such that

$$\int_{\mathbb{R}^2} u(x, t)(\partial_x^2 \varphi(x, t) - \partial_t^2 \varphi(x, t))dxdt = 0, \quad (9)$$

for any function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Show that if $f \in L^2(\mathbb{R})$, then

$$u(x, t) = f(x - t) \text{ and } v(x, t) = f(x + t) \quad (10)$$

are weak solutions of the wave equation. Hint: Approximate!

5. Let $\{w_1, \dots, w_N\}$ be distinct points in the open unit disk, and $\{a_1, \dots, a_N\}$ complex numbers. We define the affine subspace $Y_a \subset \mathcal{H}^2(D_1)$, to be

$$Y_a = \{f \in \mathcal{H}^2(D_1) : f(w_j) = a_j\}. \quad (11)$$

Show that $Y_a \neq \emptyset$. Define

$$m(\mathbf{a}) = \inf\{\|f\|_2 : f \in Y_a\}. \quad (12)$$

Prove that there is a unique function $f_1 \in Y_a$ with $m(\mathbf{a}) = \|f_1\|$.

Let $g_w \in \mathcal{H}^2(D_1)$ be the unique function so that, for all $f \in \mathcal{H}^2(D_1)$,

$$f(w) = \int_{D_1} f(z) \overline{g_w(z)} dx dy \quad (13)$$

Prove that we can choose $(\lambda_1, \dots, \lambda_N)$ so that

$$F_1 = \sum_{j=1}^N \lambda_j g_{w_j} \in Y_a. \quad (14)$$

Hint: See 1 in the “standard problems” section.

Let Y_0 be the subspace of $\mathcal{H}^2(D_1)$ consisting of functions that vanish at $\{w_j\}$. Prove $\langle F_1, F_0 \rangle = 0$ for all $F_0 \in Y_0$ and explain why this shows that

$$m(\mathbf{a}) = \sqrt{\sum_{j,k=1}^N \langle g_{w_j}, g_{w_k} \rangle \lambda_j \bar{\lambda}_k} = \sqrt{\sum_{j=1}^N a_j \bar{\lambda}_j}. \quad (15)$$

Hence F_1 is the unique function in Y_a with

$$\|F_1\|_2 = m(\mathbf{a}). \quad (16)$$