AMCS/MATH 609
Problem set 6 due March 17, 2015
Dr. Epstein

Reading: Read Chapters 3.2-3.3 (especially the proof of Theorem 8), 4.2, 5.1-5.2, 6.1-6.3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Prove Theorem 4 in §3.2 of Lax.

2. Suppose that (X, d) is a metric space. Show that if $\lim_{n\to\infty} x_n = x^*$, then, for any $x \in X$, we also have that $\lim_{n\to\infty} d(x, x_n) = d(x, x^*)$.

3. Prove that in any real normed linear space $(X, \|\cdot\|)$, the open and closed unit balls
$$B_1 = \{x \in X : \|x\| < 1\}, \quad \overline{B}_1 = \{x \in X : \|x\| \leq 1\} \tag{1}$$
are convex and have non-empty interior. The unit ball is *strictly convex*, if, whenever $\|x\| = \|y\| = 1$ and $x \neq y$, then
$$\left\| \frac{x+y}{2} \right\| < 1. \tag{2}$$
Show that the unit ball in ℓ_2 is strictly convex, but the unit ball in ℓ_1 is not.

4. Let $Y \subset \ell_\infty$ be the subspace of sequences that are eventually zero (only finitely many terms non-zero). Find the closure of Y with respect to the ℓ_∞-norm.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that every finite dimensional subspace of a normed vector space is closed. Hint: Use the fact that all norms on a finite dimensional vector space are equivalent to show that every finite dimensional subspace is complete.

2. Let V be a vector space, possibly infinite dimensional.

   (a) Show that if $\mathscr{X} = \{x_\alpha : \alpha \in \mathscr{A}\} \subset V$ is a set of linearly independent vectors, then there is a basis for V of the form $\{x_\alpha : \alpha \in \mathscr{A}\} \cup \{y_\beta : \beta \in \mathscr{B}\}$. Hint: Let $\mathscr{W}$ consists of sets of linearly independent vectors in V, with the partial ordered defined by inclusion, then apply Zorn's lemma to prove this assertion.

(b) Use this result to show that if $U \subset V$ is a subspace of V, then there exists another subspace W of V so that $V = U \oplus W$, and an isomorphism

$$\varphi : W \longrightarrow V/U. \tag{3}$$

3. [This problem assumes an elementary knowledge of holomorphic functions of one complex variable.]
$L^2(D_1)$ is defined as the closure of $C^0(\overline{D}_1)$ with respect to the L^2-norm. Note that the points of $L^2(D_1)$ are equivalence classes of functions, equal almost everywhere. The norm on $L^2(D_1)$ is

$$\| f \|_2^2 = \int_{D_1} |f(x,y)|^2 dxdy = \lim_{r \to 1^+} \iint_{D_r} |f(x,y)|^2 dxdy < \infty. \tag{4}$$

Let $\mathcal{H}^2(D_1)$ denote the closure, with respect to the L^2-norm, of holomorphic functions on the unit disk that are in $\mathcal{C}^0(\overline{D}_1)$.

(a) Show that $\mathcal{H}^2(D_1)$ is a closed subspace of $L^2(D_1)$ and that $f \in \mathcal{H}^2(D_1)$ is holomorphic in int D_1. That is, every element of $\mathcal{H}^2(D_1)$ has a representative that is holomorphic in int D_1.

(b) Show that if f is a square integrable function in D_1, which is holomorphic in the interior of D_1, then $f \in \mathcal{H}^2(D_1)$. (You need to show that f is an L^2-limit of functions in $C^0(\overline{D}_1)$.

(c) Give an orthonormal basis for $\mathcal{H}^2(D_1)$. Let $P : L^2(D_1) \to \mathcal{H}^2(D_1)$ denote the nearest neighbor map, or orthogonal projection onto $\mathcal{H}^2(D_1)$. Find an expression for Pf in terms of the orthonormal basis. Can you prove that

$$Pf(z) = \int_{D_1} \frac{f(w,\bar{w})dxdy}{\pi(1 - z\bar{w})^2}? \tag{5}$$

(d) Prove that for any $k \in \mathbb{N}$, there is a bounded linear functional ℓ_k defined on $L^2(D_1)$, so that if $f \in \mathcal{H}^2(D_1)$, then

$$\ell_k(f) = \partial_z^k f(0). \tag{6}$$

4. A bounded sequence $< c_j >$ is Cesaro summable if

$$\lim_{n \to \infty} \frac{c_1 + \cdots + c_n}{n} \text{ exists.} \tag{7}$$

Show that a Banach limit LIM can be defined on ℓ_∞ so that if $< c_j >$ is Cesaro summable then

$$\operatorname*{LIM}_{j\to\infty} c_j = \lim_{n\to\infty} \frac{c_1 + \cdots + c_n}{n}. \tag{8}$$

5. Let $\mathcal{P}$ denote the subspace of $\mathcal{C}^0([0,1])$ defined by polynomials restricted to [0, 1]. Suppose that $\ell : \mathcal{P} \to \mathbb{R}$ is a linear function with the property that

$$p(x) \geq 0 \text{ for } x \in [0,1] \Rightarrow \ell(p) \geq 0. \tag{9}$$

Show that ℓ extends to define a linear functional , $\tilde{\ell}$, on all of $\mathcal{C}^0([0,1])$, satisfying an estimate of the form

$$|\tilde{\ell}(f)| \leq C\|f\|_\infty. \tag{10}$$

Can you find a closed form expression for C?

6. Prove that ℓ_1 has a countable dense subset, but ℓ_∞ does not. Recall that:

$$\ell_1 = \left\{(a_1, a_2, \dots) : \|a\|_1 = \sum_j |a_j| < \infty\right\}$$

and

$$\ell_\infty = \left\{(a_1, a_2, \dots) : \|a\|_\infty = \sup\{|a_j| : j = 1, 2, \dots\} < \infty\right\}.$$