

# AMCS 609

Problem set 6 due March 15, 2011

Dr. Epstein

**Reading:** Read Chapters 8.1-3, and 9.1 in Lax, *Functional Analysis*. **Standard problem:** The following problem should be done, but does not have to be handed in.

1. Exercise 1 on page 76 of Lax.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Review the definition of a *Banach Limit*, on pages 31-2 of Lax. As shown there,  $\text{LIM}_{n \rightarrow \infty} a_n$  is a linear functional on bounded sequences with

$$\liminf_{n \rightarrow \infty} a_n \leq \text{LIM}_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n. \quad (1)$$

Show that this implies that  $\text{LIM}_{n \rightarrow \infty} \in \ell'_\infty$ , but there does **not** exist a vector  $\mathbf{b} = (b_1, b_2, \dots) \in \ell_1$ , such that

$$\text{LIM}_{n \rightarrow \infty} a_n = \sum_{n=1}^{\infty} a_n b_n. \quad (2)$$

2. Exercise 2 on page 76 of Lax.
3. Exercise 3 on page 77 of Lax.
4. Exercise 5 on page 80 of Lax.
5. Prove that  $L^2([0, 1])$ , the closure of  $C^0([0, 1])$  with respect to

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 dx, \quad (3)$$

is a separable space.

6. Suppose that  $\ell$  is a bounded linear functional on a Hilbert space, and  $\{e_j\}$  is a collection of orthonormal vectors. Show that

$$\lim_{j \rightarrow \infty} \ell(e_j) = 0. \quad (4)$$