

AMCS 609

Problem set 7 due March 26, 2009

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Reading: Read Chapters 15 and 16 in Lax, *Functional Analysis*.

Standard problem: The following problem should be done, but do not have to be handed in.

1. Let $M : X \rightarrow Y$ be linear. Show that the range of M is dense if and only if $\ker M' = \{0\}$.
2. Lax page 165, exercise 3.
3. Lax page 166, exercise 7.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. If $1 \leq p < q < \infty$, then $\ell_p \subset \ell_q$. For fixed $p < q$, and $n \in \mathbb{N}$, show that the set

$$B_n = \{(x_j) \in \ell_q : \sum_{j=1}^{\infty} |x_j|^p \leq n\} \quad (1)$$

is closed and nowhere dense, as a subset of ℓ_q . Hence, as a subset of ℓ_q , ℓ_p is a set of first category.

2. Consider the interval $[0, 1]$ as a complete metric space with $d(x, y) = |x - y|$.
 - (a) Find a subset $S_1 \subset [0, 1]$ of first category which is also dense. So sets of the first category, which are in some sense small, can also be, in another sense, large.
 - (b) Find a subset $S_2 \subset [0, 1]$ of second category with measure 0. This means that for any $\epsilon > 0$, there is a cover of S_2 by open intervals $\{(a_j, b_j)\}$ for which

$$\sum_{j=1}^{\infty} (b_j - a_j) < \epsilon. \quad (2)$$

So sets of the second category, which are in some sense large, can also be, in another sense, small.

3. Let X be a Banach space and $T \in \mathcal{L}(X, X)$ with $|T| < 1$.

(a) Prove that $S_n = \sum_{j=0}^n T^j$ is a norm convergent sequence in $\mathcal{L}(X, X)$. Let S denote the limit.

(b) Prove that for every $y \in X$ we have

$$(\text{Id} - T)Sy = y \tag{3}$$

and conclude that $(\text{Id} - T)$ is boundedly invertible. Give an estimate for $|S|$ in terms of $|T|$.

(c) Show that if $M \in \mathcal{L}(X, X)$ is invertible, then there is an $\epsilon > 0$ so that every $N \in \mathcal{L}(X, X)$ with $|M - N| < \epsilon$ is also invertible. Briefly, invertibility is an open property in the operator topology.

(d) If $M \in \mathcal{L}(X, X)$, then we define the resolvent set of M to be

$$\rho(M) = \{\lambda \in \mathbb{C} : (M - \lambda \text{Id}) \text{ is invertible} \}. \tag{4}$$

Prove that $\rho(M)$ is an open subset of \mathbb{C} .

4. Suppose that X, Y are Banach spaces and $M : X \rightarrow Y$ is a surjective, bounded linear map. Show that there is a constant $c > 0$, so that for every $y \in Y$, there exists an $x \in X$ with

$$Mx = y \text{ and } \|x\| < c\|y\|. \tag{5}$$

5. Let X, Y, W be Banach spaces, with sequences $\langle S_n \rangle \in \mathcal{L}(Y, W)$, $\langle T_n \rangle \in \mathcal{L}(X, Y)$.

(a) If $S_n \rightarrow S$ and $T_n \rightarrow T$ strongly, then $S_n T_n \rightarrow ST$ strongly.

(b) Suppose that S_n converges weakly to S and T_n converges strongly to T ; show that $S_n T_n$ converges weakly to ST .

(c) Find examples of $\langle S_n \rangle$, $\langle T_n \rangle$ both of which converge weakly to zero, but such that $S_n T_n$ does not. Hint: Look at the shift operator on bi-infinite square summable sequences: $S(x_j) = (x_{j+1})$.

6. Let H be a Hilbert space with $\{u_n\}$ an orthonormal basis.

(a) Define $T_k : H \rightarrow H$ by

$$T_k\left(\sum_{j=1}^{\infty} a_j u_j\right) = a_k u_k. \quad (6)$$

Prove that T_k converges to 0 in the strong sense, but not in the operator norm.

(b) Define $S_k : H \rightarrow H$ by

$$S_k\left(\sum_{j=1}^{\infty} a_j u_j\right) = \sum_{j=1}^{\infty} a_j u_{j+k}. \quad (7)$$

Show that S_k converges to 0 in the weak sense, but not in the strong sense.