

## AMCS 609

Problem set 7 due March 22, 2010

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**Reading:** Read Chapters 10 and 11 in Lax, *Functional Analysis*. You might also want to look at the sections in Royden, *Real Analysis*, or Rudin, *Real and Complex Analysis* on the Baire Category Theorem and the Uniform Boundedness Principle.

**Standard problem:** The following problems should be done, but do not have to be handed in.

1. Let  $(X, d)$  be a metric space and  $U, V$  open dense subsets of  $X$ . Show that  $U \cap V$  is also dense.
2. Exercise 3 on page 101 of Lax.
3. Exercise 5 on page 104 of Lax.
4. Exercise 6 on page 106 of Lax.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Prove: If  $\langle x_n \rangle$  is a sequence in  $\ell_1$  that converges weakly to 0, then

$$\lim_{n \rightarrow \infty} \|x_n\|_1 = 0, \quad (1)$$

that is:  $\langle x_n \rangle$  also converges strongly to zero. Hints: Argue by contradiction, choose an appropriate subsequence, and use the fact that  $\ell_1^* = \ell_\infty$  is a very big vector space.

2. Suppose that  $\langle b_j \rangle$  is a sequence of real numbers so that, for every real sequence  $\langle a_j \rangle$ , converging to zero, the limit

$$\ell(\mathbf{a}) = \lim_{N \rightarrow \infty} \sum_{j=1}^N a_j b_j \quad (2)$$

exists. Prove that

$$\sum_{j=1}^{\infty} |b_j| < \infty. \quad (3)$$

3. Let  $(a_{ij})$  be an infinite matrix with complex entries,  $1 \leq i, j < \infty$ . Suppose that for every convergent sequence  $\langle s_j \rangle$ , and  $1 \leq i$ , we define

$$\sigma_i = \lim_{N \rightarrow \infty} \sum_{j=1}^N a_{ij} s_j, \quad (4)$$

if the limit exists.

Show that these limits exist, for all convergent sequences  $\langle s_j \rangle$ , and define a sequence  $\langle \sigma_i \rangle$ , with the same limit, if and only if the following conditions hold:

(a)

$$\lim_{i \rightarrow \infty} a_{ij} = 0 \text{ for each } j.$$

(b)

$$\sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty.$$

(c)

$$\lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} a_{ij} = 1.$$

Give an example of such a matrix for which there exists a non-convergent sequence,  $\langle s_j \rangle$ , so that  $\sigma_j$  exists for every  $j \in \mathbb{N}$ , and the sequence  $\langle \sigma_j \rangle$  is convergent.

4. Let  $(X, \|\cdot\|)$ , be a normed linear space, and  $X'$  its dual space, with norm defined by

$$|\ell| = \sup_{x \neq 0} \frac{|\ell(x)|}{\|x\|}. \quad (5)$$

Let  $\{x_n\}$  be a sequence of points in  $X$  for which  $\{\ell(x_n)\}$  is a bounded sequence, for every  $\ell \in X'$ . Prove that  $\{\|x_n\|\}$  is also bounded.

5. Let  $\{f_n\}$  be a sequence of continuous, real valued functions defined on  $[0, 1]$ , such that  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists for every  $x \in [0, 1]$ .

(a) Prove that there is a non-empty open set  $V \subset [0, 1]$ , and a number  $M$  such that

$$|f_n(x)| < M \text{ for all } x \in V. \quad (6)$$

(b) If  $\epsilon > 0$ , show that there is a nonempty open set,  $V$  and an integer  $N$  so that

$$|f(x) - f_n(x)| < \epsilon \text{ for all } x \in V. \quad (7)$$

Hint: For each  $N$  define  $A_N = \{x : |f_n(x) - f_m(x)| \leq \epsilon \text{ if } N \leq n, m\}$ , and consider  $\cup_N A_N$ .

6. If  $1 \leq p < q < \infty$ , then  $\ell_p \subset \ell_q$ . For fixed  $p < q$ , and  $n \in \mathbb{N}$ , show that the set

$$B_n = \{(x_j) \in \ell_q : \sum_{j=1}^{\infty} |x_j|^p \leq n\} \quad (8)$$

is closed and nowhere dense, as a subset of  $\ell_q$ . Hence, as a subset of  $\ell_q$ ,  $\ell_p$  is a set of first category.

7. Consider the interval  $[0, 1]$  as a complete metric space with  $d(x, y) = |x - y|$ .

(a) Find a subset  $S_1 \subset [0, 1]$  of first category which is also dense. So sets of the first category, which are in some sense small, can also be, in another sense, large.

(b) Find a subset  $S_2 \subset [0, 1]$  of second category with measure 0. This means that for any  $\epsilon > 0$ , there is a cover of  $S_2$  by open intervals  $\{(a_j, b_j)\}$  for which

$$\sum_{j=1}^{\infty} (b_j - a_j) < \epsilon. \quad (9)$$

So sets of the second category, which are in some sense large, can also be, in another sense, small.