# AMCS/MATH 609
# Problem set 7 due March 31, 2015
# Dr. Epstein

**Reading:** Read Chapters 5, 8.1-3, and 9.1 in Lax, *Functional Analysis*. **Standard problem:** The following problems should be done, but do not have to be handed in.

1. Exercises 1 and 2 on page 76 of Lax.

2. Let $\{V_1, \ldots, V_N\}$ be linearly independent vectors in a complex inner product space $(X, \langle \cdot, \cdot \rangle)$.

   (a) Show that the matrix
   $$G_{ij} = \langle V_i, V_j \rangle, \tag{1}$$
   is Hermitian symmetric that is $G_{ij} = \overline{G_{ji}}$.

   (b) Prove that for $(v_1, \ldots, v_n),\ (w_1, \ldots, w_N) \in \mathbb{C}^N$, we have
   $$\left| \sum_{i,j=1}^{N} G_{ij} v_i \overline{w}_j \right| \leq \sqrt{\sum_{i,j=1}^{N} G_{ij} v_i \overline{v}_j} \sqrt{\sum_{i,j=1}^{N} G_{ij} w_i \overline{w}_j} \tag{2}$$

   (c) Without using computation, provide a conceptual proof that $G_{ij}$ positive definite, i.e., there is a positive constant $C$ so that for $(v_1, \ldots, v_n) \in \mathbb{C}^N$, we have
   $$\sum_{i,j=1}^{N} G_{ij} v_i \overline{v}_j \geq C \sum_{j=1}^{N} |v_j|^2. \tag{3}$$

   (d) Show that $G_{ij}$ is invertible.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Suppose that $X$ is a Banach space and $Y \subset X$ is a closed subspace. Show that the quotient space $X/Y$, with the quotient norm
   $$\|[x]\|_{X/Y} = \inf_{x \in [x]} \|x\|_X, \tag{4}$$
   is complete.

2. A bounded sequence $< c_j >$ is Cesaro summable if

$$\lim_{n\to\infty} \frac{c_1 + \cdots + c_n}{n} \text{ exists.} \tag{5}$$

Show that a Banach limit LIM can be defined on $\ell_\infty$ so that if $< c_j >$ is Cesaro summable then

$$\operatorname*{LIM}_{j\to\infty} c_j = \lim_{n\to\infty} \frac{c_1 + \cdots + c_n}{n}. \tag{6}$$

3. Let LIM be as defined in the previous exercise. Show that there does not exist a vector $\boldsymbol{b} \in \ell_1$ so that

$$\operatorname*{LIM}_{j\to\infty} c_j = \langle \boldsymbol{c}, \boldsymbol{b} \rangle. \tag{7}$$

4. Let $\{w_1, \ldots, w_N\}$ be distinct points in the open unit disk, and $\{a_1, \ldots, a_N\}$ complex numbers. We define the affine subspace $Y_{\boldsymbol{a}} \subset \mathcal{H}^2(D_1)$, to be

$$Y_{\boldsymbol{a}} = \{ f \in \mathcal{H}^2(D_1) : \ f(w_j) = a_j \}. \tag{8}$$

Show that $Y_{\boldsymbol{a}} \neq \emptyset$. Define

$$m(\boldsymbol{a}) = \inf\{\| f\|_2 : \ f \in Y_{\boldsymbol{a}}\}. \tag{9}$$

Prove that there is a unique function $f_1 \in Y_{\boldsymbol{a}}$ with $m(\boldsymbol{a}) = \| f_1\|$.

Let $g_w \in \mathcal{H}^2(D_1)$ be the unique function so that, for all $f \in \mathcal{H}^2(D_1)$,

$$f(w) = \iint\limits_{D_1} f(z)\overline{g_w(z)} dx dy \tag{10}$$

Prove that we can choose $(\lambda_1, \ldots, \lambda_N)$ so that

$$F_1 = \sum_{j=1}^{N} \lambda_j g_{w_j} \in Y_{\boldsymbol{a}}. \tag{11}$$

Let $Y_0$ be the subspace of $\mathcal{H}^2(D_1)$ consisting of functions that vanish at $\{w_j\}$. Prove $< F_1, F_0 >= 0$ for all $F_0 \in Y_0$ and explain why this shows that

$$m(\boldsymbol{a}) = \sqrt{\sum_{j,k=1}^{N} \langle g_{w_j}, g_{w_k} \rangle \lambda_j \overline{\lambda}_k} = \sqrt{\sum_{j=1}^{N} a_j \overline{\lambda}_j}. \tag{12}$$

Hence $F_1$ is the unique function in $Y_{\boldsymbol{a}}$ with

$$\|F_1\|_2 = m(\boldsymbol{a}). \tag{13}$$

5. Suppose that $f \in C^0([0,1])$ and for every function $\varphi \in \mathscr{C}_0^\infty((0,1))$, we have

$$\int_0^1 f(x)\varphi(x)dx = 0, \tag{14}$$

prove that $f = 0$ in $C^0([0,1])$. Now show that if $f \in L^2([0,1])$ and this condition holds for all $\varphi \in \mathscr{C}_0^\infty((0,1))$, then $f = 0$ in $L^2([0,1])$. Remember that $L^2([0,1])$ is the closure of $C^0([0,1])$ with respect to the $L^2$-norm. The proofs are completely different in the two cases!

6. A function $u$, which belongs to $L^2([-R,R])$ for all $R > 0$, is weakly constant if

$$\int u(x)\partial_x\varphi(x) = 0 \tag{15}$$

for every $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$. Show that a weakly constant function is smooth and constant, or more accurately: has a smooth representative, which is constant. Hint: Show that if $u(x)$ is weakly constant then so is $au(x-y)$ for all $a, y \in \mathbb{R}$.

7. Let $D$ be a connected open subset of $\mathbb{C}$. A function $f \in L^2(D)$ is weakly holomorphic if

$$\iint_D f\partial_{\bar{z}}\varphi dxdy = 0, \tag{16}$$

for all $\varphi \in \mathscr{C}_c^\infty(D)$. Show that a weakly holomorphic function is smooth in the interior of $D$ and is a classical solution to the PDE $\partial_{\bar{z}} f = 0$. Note: Use the definition of weakly holomorphic given here, which differs from that given last semester. In particular, a weakly holomorphic function is not known, a priori, to be continuous.

8. Let $Y = \{u \in \mathscr{C}^\infty(\overline{D}_1) : \Delta u = 0\}$.

   (a) We let $\overline{Y}$ denote the closure of $Y$ with respect to the $L^2$-norm on the unit disk:

   $$\|u\|_2^2 = \iint_{D_1} |u(x,y)|^2 dxdy. \tag{17}$$

   Show that if $v \in \overline{Y}$ then $v$ has representative that is smooth in the interior of the unit disk and that $\Delta v = 0$, in the interior of $D_1$. Hint: Use the Poisson formula.

   (b) Describe the radial functions in $Y^\perp$, that is functions of $r = \sqrt{x^2+y^2}$ that are orthogonal to $\overline{Y}$.