

AMCS 609

Problem set 8 due April 2, 2009

Dr. Epstein

Reading: Read Chapters 15 and 16 in Lax, *Functional Analysis*.

Standard problem: The following problem should be done, but do not have to be handed in.

1. Lax page 168, exercise 8.
2. Lax page 168, exercise 9.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $S \subset C^0([0, 1])$, which is closed with respect to the L^2 -norm. This means that if $\langle f_n \rangle \subset S$, and there is a function $f \in L^2[0, 1]$ such that $\|f_n - f\|_{L^2} \rightarrow 0$, then f can be represented by a function in S .
 - (a) Show that S is also closed as a subspace of C^0 , hint: use the closed graph theorem.
 - (b) Show that there is a constant M so that, for $f \in S$, we have

$$\|f\|_\infty < M\|f\|_2. \quad (1)$$

- (c) Show that for each $y \in [0, 1]$ there is a function $k_y \in L^2([0, 1])$ so that

$$f(y) = \int_0^1 f(x)k_y(x)dx. \quad (2)$$

2. Let X be a separable Banach space with $\{x_n\}$ a countable dense subset of the unit ball. We define a map $T : \ell_1 \rightarrow X$, by setting:

$$T(\mathbf{a}) = \sum_{j=1}^{\infty} a_j x_j. \quad (3)$$

- (a) Prove that T is bounded.

- (b) Prove that T is surjective. Hint: you should find a direct argument.
- (c) Show that X is isomorphic to a quotient space of ℓ_1 .
3. (a) Suppose that X and Y are Banach spaces, and $D \subset X$ is a linear subspace, which may not be closed. Suppose that $T : D \rightarrow Y$ has a closed graph, and is 1-1 and onto. If D is not closed, then T need not be continuous. Prove, however, that $T^{-1} : Y \rightarrow X$ is continuous.
- (b) Let X denote continuous functions on $[0, 1]$ that vanish at 0; $Y = C^0([0, 1])$; and $D \subset X$, those functions with a continuous first derivative. Show that $Tf = \partial_x f$ has a closed graph, and is a 1-1, onto map from D to Y . What is T^{-1} ? Give an elementary proof that it is bounded as a map from $Y \rightarrow X$.
4. Let X be a Banach space and $T : X \rightarrow X$ a linear map. Suppose that there is a continuous 1-1 linear map $L : X \rightarrow X$, such that LT is continuous. Does this imply that T is continuous?