AMCS/MATH 609
Problem set 8
Due April 7, 2015
Dr. Epstein

Reading: Read Chapters 8.1-3, 9.1, and 10.1-3 in Lax, *Functional Analysis*.
Standard problems: The following problems should be done, but do not have to be handed in:

1. Let $Y \subset X$, a normed linear space. Show that $Y^{\perp}$ is a closed subspace of X'.

2. Prove that $L^2([0,1])$, the closure of $C^0([0,1])$ with respect to

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 dx, \tag{1}$$

is a separable space.

3. Suppose that $(M, \Sigma, d\mu)$ is a measure space, with $\mu(M) = 1$. Show that

$$\|f\|_p = \left[\int_M |f(m)|^p d\mu(m)\right]^{\frac{1}{p}} \tag{2}$$

is an increasing function of p.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. For $f \in \mathscr{C}_0^{\infty}(\mathbb{R})$, show that $|f|$ has a weak derivative, which can be represented by a function $g(x)$ that satisfies

$$|g(x)| \leq |\partial_x f(x)|. \tag{3}$$

The statement that the weak derivative is *represented by g* means that for all $\varphi \in \mathscr{C}_0^{\infty}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} |f(x)| \partial_x \varphi(x) dx = -\int_{\mathbb{R}} g(x)\varphi(x) dx. \tag{4}$$

Hint: Consider $\sqrt{f^2(x) + \epsilon^2}$. Compute the weak derivative of $|x|$.

2. Suppose that we define a weak solution of the wave equation,

$$\partial_x^2 u(x,t) - \partial_t^2 u(x,t) = 0, \tag{5}$$

to be a function that is square integrable in $[-R, R] \times [-R, R]$ for any R, and such that

$$\int_{\mathbb{R}^2} u(x,t)(\partial_x^2 \varphi(x,t) - \partial_t^2 \varphi(x,t))dxdt = 0, \tag{6}$$

for any function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. Show that if $f \in L^2(\mathbb{R})$, then

$$u(x,t) = f(x-t) \text{ and } v(x,t) = f(x+t) \tag{7}$$

are weak solutions of the wave equation. Hint: Approximate!

3. The space $H_1(D_1)$ is the closure of $\mathscr{C}^\infty(\overline{D_1})$ with respect to the norm

$$\|u\|_1^2 = \int_{D_1} [|u(x)|^2 + |\nabla u(x)|^2] dx. \tag{8}$$

In class we proved that the map $R : u \mapsto u \restriction_{\partial D_1}$ has a continuous extension as a map $R : H_1(D_1) \to L^2(\partial D_1)$. Prove that there are functions $f \in L^2(\partial D_1)$ for which there does **not** exist a function $u \in H_1(D_1)$ for which $f = R(u)$. Hint: The argument given in class actually showed that $R(u)$ belongs to a subspace of $L^2(\partial D_1)$.

4. Let $Y \subset X$ be a closed subspace of a normed linear space. Show that

$$Y^\perp = \{\ell \in X' | \ell(y) = y \text{ for all } y \in Y\} \tag{9}$$

is isometrically isomorphic to $(X/Y)'$. Show that Y' is isometrically isomorphic to $X'/Y^\perp$.

5. Suppose that ℓ is a bounded linear functional on a Hilbert space, and $\{e_j\}$ is a collection of orthonormal vectors. Show that

$$\lim_{j \to \infty} \ell(e_j) = 0. \tag{10}$$