

AMCS 609

Problem set 8 due April 5, 2011

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**Reading:** Read Chapters 15 and 16 in Lax, *Functional Analysis*.

**Standard problem:** The following problem should be done, but do not have to be handed in.

1. Let  $X, Y, W$  be Banach spaces, with sequences  $\langle S_n \rangle \in \mathcal{L}(Y, W)$ ,  $\langle T_n \rangle \in \mathcal{L}(X, Y)$ .
  - (a) If  $S_n \rightarrow S$  and  $T_n \rightarrow T$  strongly, then  $S_n T_n \rightarrow ST$  strongly.
  - (b) Suppose that  $S_n$  converges weakly to  $S$  and  $T_n$  converges strongly to  $T$ ; show that  $S_n T_n$  converges weakly to  $ST$ .
  - (c) Find examples of  $\langle S_n \rangle$ ,  $\langle T_n \rangle$  both of which converge weakly to zero, but such that  $S_n T_n$  does not. Hint: Look at the shift operator on bi-infinite square summable sequences:  $S(x_j) = (x_{j+1})$ .
2. Lax page 165, exercise 3.
3. Lax page 166, exercise 7.
4. Lax page 168, exercise 8.
5. Lax page 168, exercise 9.
6. Lax page 172, exercise 13.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Let  $M : X \rightarrow Y$  be linear. Show that the range of  $M$  is dense if and only if  $\ker M' = \{0\}$ .
2. Let  $X$  be a Banach space and  $T \in \mathcal{L}(X, X)$  with  $|T| < 1$ .
  - (a) Prove that  $S_n = \sum_{j=0}^n T^j$  is a norm convergent sequence in  $\mathcal{L}(X, X)$ . Let  $S$  denote the limit.

(b) Prove that for every  $y \in X$  we have

$$(\text{Id} - T)Sy = y \quad (1)$$

and conclude that  $(\text{Id} - T)$  is boundedly invertible. Give a estimate for  $|S|$  in terms of  $|T|$ .

(c) Show that if  $M \in \mathcal{L}(X, X)$  is invertible, then there is an  $\epsilon > 0$  so that every  $N \in \mathcal{L}(X, X)$  with  $|M - N| < \epsilon$  is also invertible. Briefly, invertibility is an open property in the operator topology.

(d) If  $M \in \mathcal{L}(X, X)$ , then we define the resolvent set of  $M$  to be

$$\rho(M) = \{\lambda \in \mathbb{C} : (M - \lambda \text{Id}) \text{ is invertible} \}. \quad (2)$$

Prove that  $\rho(M)$  is an open subset of  $\mathbb{C}$ .

3. Suppose that  $X, Y$  are Banach spaces and  $M : X \rightarrow Y$  is a surjective, bounded linear map. Show that there is a constant  $c > 0$ , so that for every  $y \in Y$ , there exists an  $x \in X$  with

$$Mx = y \text{ and } \|x\| < c\|y\|. \quad (3)$$

4. Let  $H$  be a Hilbert space with  $\{u_n\}$  an orthonormal basis.

(a) Define  $T_k : H \rightarrow H$  by

$$T_k\left(\sum_{j=1}^{\infty} a_j u_j\right) = a_k u_k. \quad (4)$$

Prove that  $T_k$  converges to 0 in the strong sense, but not in the operator norm.

(b) Define  $S_k : H \rightarrow H$  by

$$S_k\left(\sum_{j=1}^{\infty} a_j u_j\right) = \sum_{j=1}^{\infty} a_j u_{j+k}. \quad (5)$$

Show that  $S_k$  converges to 0 in the weak sense, but not in the strong sense.

5. Let  $X$  be a separable Banach space with  $\{x_n\}$  a countable dense subset of the unit ball. We define a map  $T : \ell_1 \rightarrow X$ , by setting:

$$T(\mathbf{a}) = \sum_{j=1}^{\infty} a_j x_j. \quad (6)$$

- (a) Prove that  $T$  is bounded.
- (b) Prove that  $T$  is surjective. Hint: you should find a direct argument.
- (c) Show that  $X$  is isomorphic to a quotient space of  $\ell_1$ .
6. Let  $S \subset C^0([0, 1])$ , which is closed with respect to the  $L^2$ -norm. This means that if  $\langle f_n \rangle \subset S$ , and there is a function  $f \in L^2[0, 1]$  such that  $\|f_n - f\|_{L^2} \rightarrow 0$ , then  $f$  can be represented by a function in  $S$ .

- (a) Show that  $S$  is also closed as a subspace of  $C^0$ .
- (b) Show that there is a constant  $M$  so that, for  $f \in S$ , we have

$$\|f\|_\infty < M\|f\|_2. \quad (7)$$

Hint: use the closed graph theorem.

- (c) Show that for each  $y \in [0, 1]$  there is a function  $k_y \in L^2([0, 1])$  so that

$$f(y) = \int_0^1 f(x)k_y(x)dx. \quad (8)$$

7. (a) Suppose that  $X$  and  $Y$  are Banach spaces, and  $D \subset X$  is a linear subspace, which may not be closed. Suppose that  $T : D \rightarrow Y$  has a closed graph, and is 1-1 and onto. If  $D$  is not closed, then  $T$  need not be continuous. Prove, however, that  $T^{-1} : Y \rightarrow X$  is continuous.
- (b) Let  $X$  denote continuous functions on  $[0, 1]$  that vanish at 0;  $Y = C^0([0, 1])$ ; and  $D \subset X$ , those functions with a continuous first derivative. Show that  $Tf = \partial_x f$  has a closed graph, and is a 1-1, onto map from  $D$  to  $Y$ . What is  $T^{-1}$ ? Give an *elementary* proof that it is bounded as a map from  $Y \rightarrow X$ .
8. Let  $X$  be a Banach space and  $T : X \rightarrow X$  a linear map. Suppose that there is a continuous 1-1 linear map  $L : X \rightarrow X$ , such that  $LT$  is continuous. Does this imply that  $T$  is continuous?
9. Suppose that  $k(s, t)$  is a measurable function on  $S \times T$  such that

$$M_1 = \sup_{s \in S} \int_T |k(s, t)|dn(t) < \infty \text{ and} \quad (9)$$

$$M_2 = \sup_{t \in T} \int_S |k(s, t)|dm(s) < \infty.$$

Show that for every  $1 < p < \infty$  the operator

$$Kf(s) = \int_T k(s, t) f(t) dn(t) \quad (10)$$

is bounded from  $L^p(T; dn) \rightarrow L^p(S; dm)$  with  $\|K\|_{L^p \rightarrow L^p} \leq M_1^{\frac{1}{q}} M_2^{\frac{1}{p}}$ . Here  $p^{-1} + q^{-1} = 1$ . Show that if

$$\int_{-\infty}^{\infty} |\phi(x)| dx < \infty, \quad (11)$$

then the operator

$$K_\phi f(x) = \int_{-\infty}^{\infty} \phi(x - y) f(y) dy \quad (12)$$

is bounded from  $L^2(\mathbb{R})$  to itself.