

# AMCS 609

Problem set 9 due April 9, 2009

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**Reading:** Read Chapters 16 and 21 in Lax, *Functional Analysis*.

**Standard problem:** The following problems should be done, but do not have to be handed in.

1. Lax page 182, exercise 1.
2. Lax page 184, exercise 3.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Suppose that  $k(s, t)$  is a measurable function on  $S \times T$  such that

$$\begin{aligned} M_1 &= \sup_{s \in S} \int_T |k(s, t)| dn(t) < \infty \text{ and} \\ M_2 &= \sup_{t \in T} \int_S |k(s, t)| dm(s) < \infty. \end{aligned} \tag{1}$$

Show that for every  $1 < p < \infty$  the operator

$$Kf(s) = \int_T k(s, t) f(t) dn(t) \tag{2}$$

is bounded from  $L^p(T; dn) \rightarrow L^p(S; dm)$  with  $\|K\|_{L^p \rightarrow L^p} \leq M_1^{\frac{1}{q}} M_2^{\frac{1}{p}}$ . Here  $p^{-1} + q^{-1} = 1$ .

2. Prove that the operator:

$$Kf(x) = \int_{-\infty}^{\infty} e^{-|x-y|} f(y) dy, \tag{3}$$

is a bounded operator on  $L^2(\mathbb{R})$ .

3. For  $\alpha > 0$ , define an integral operator

$$K_\alpha f(x) = \int_0^x \frac{f(y)dy}{(x-y)^{1-\alpha}}. \quad (4)$$

- (a) Show that  $K_\alpha$  defines a continuous map from  $C^0[0, 1]$  to itself.
- (b) Show that  $K_\alpha$  defines a continuous map of  $L^p[0, 1]$  to itself for every  $1 \leq p < \infty$ .
- (c) Define the function

$$B(\alpha, \beta) = \int_0^1 \frac{du}{(1-u)^{1-\alpha}u^{1-\beta}}. \quad (5)$$

Show that  $K_\alpha \circ K_\beta = B(\alpha, \beta)K_{\alpha+\beta}$ .

- (d) What happens in the previous part if  $0 < \alpha < 1$ , and we set  $\beta = 1-\alpha$ ? Use this to find a formula for  $K_\alpha^{-1}$ . Is this operator bounded from  $L^p[0, 1] \rightarrow L^p[0, 1]$  for any  $1 \leq p \leq \infty$ ?
4. Recall that we define the Fourier transform on  $L^2(\mathbb{R})$ , by first defining it via an integral for  $f \in C_c^0(\mathbb{R})$ , then using the density of  $C_c^0(\mathbb{R})$  in  $L^2(\mathbb{R})$ , and the Parseval formula

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi \quad (6)$$

to extend it as a bounded map from  $L^2(\mathbb{R})$  to itself. Let  $f \in L^2(\mathbb{R})$ ; for  $R > 0$  define

$$\hat{f}_R(\xi) = \int_{-R}^R f(x)e^{-ix\xi} dx. \quad (7)$$

- (a) Prove that  $\lim_{R \rightarrow \infty} \hat{f}_R = \hat{f}$ , that is  $\langle \hat{f}_R \rangle$  converges in the  $L^2$ -sense to  $\hat{f}$ .
- (b) Suppose that for any  $\delta > 0$ , the functions  $\langle \hat{f}_R(\xi) \rangle$  converge uniformly to  $g(\xi)$  for  $\delta \leq |\xi| \leq \delta^{-1}$ . Prove that  $g = \hat{f}$  almost everywhere.
- (c) Prove that if  $f(x) = (x + i\epsilon)^{-1}$  for an  $\epsilon > 0$ , then

$$\hat{f}(\xi) = -2\pi i \chi_{[0, \infty)}(\xi) e^{-\epsilon\xi}. \quad (8)$$

5. Suppose that  $\varphi \in L^1(\mathbb{R})$ . Show that the integral operator defined for  $f \in C_c^0(\mathbb{R})$  by

$$\Phi f(x) = \int_{-\infty}^{\infty} \varphi(x-y)f(y)dy \quad (9)$$

extends to define a bounded map,  $\Phi : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  for all  $1 \leq p \leq \infty$ .

6. Let  $k(s, t) \in C^0([0, 1] \times [0, 1])$ , and, for  $f \in C^0([0, 1])$  define

$$Kf(s) = \int_0^s k(s, t)f(t)dt. \quad (10)$$

- (a) Prove that  $K : C^0([0, 1]) \rightarrow C^0([0, 1])$  is continuous.

- (b) Let  $M = \|k\|_{\infty}$ . Show that, for  $s \in [0, 1]$ ,

$$|Kf(s)| \leq Ms\|f\|_{\infty}. \quad (11)$$

- (c) For each  $n \in \mathbb{N}$  show that, for  $s \in [0, 1]$ ,

$$|K^n f(s)| \leq \frac{(Ms)^n}{n!} \|f\|_{\infty}. \quad (12)$$

- (d) Prove that  $(\text{Id} - K) : C^0([0, 1]) \rightarrow C^0([0, 1])$  is invertible and the inverse is given by the **norm** convergent series:

$$(\text{Id} - K)^{-1} = \sum_{n=0}^{\infty} K^n. \quad (13)$$