

AMCS 609

Problem set 9 due April 12, 2011

Dr. Epstein

Reading: Read Chapters 16 and 21 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Lax page 182, exercise 1.
2. Lax page 184, exercise 3.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. For $\alpha > 0$, define an integral operator

$$K_\alpha f(x) = \int_0^x \frac{f(y)dy}{(x-y)^{1-\alpha}}. \quad (1)$$

- (a) Show that K_α defines a continuous map from $C^0[0, 1]$ to itself.
- (b) Show that K_α defines a continuous map of $L^p[0, 1]$ to itself for every $1 \leq p < \infty$.
- (c) Define the function

$$B(\alpha, \beta) = \int_0^1 \frac{du}{(1-u)^{1-\alpha}u^{1-\beta}}. \quad (2)$$

Show that $K_\alpha \circ K_\beta = B(\alpha, \beta)K_{\alpha+\beta}$.

- (d) What happens in the previous part if $0 < \alpha < 1$, and we set $\beta = 1 - \alpha$? Use this to find a formula for K_α^{-1} . Can you give a class of functions for which the formula for K_α^{-1} makes sense?
2. Recall that we define the Fourier transform on $L^2(\mathbb{R})$, by first defining it via an integral for $f \in C_c^0(\mathbb{R})$, then using the density of $C_c^0(\mathbb{R})$ in $L^2(\mathbb{R})$, and the Parseval formula

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi \quad (3)$$

to extend it as a bounded map from $L^2(\mathbb{R})$ to itself. Let $f \in L^2(\mathbb{R})$; for $R > 0$ define

$$\hat{f}_R(\xi) = \int_{-R}^R f(x)e^{-ix\xi} dx. \quad (4)$$

- (a) Prove that l. i. m. $\hat{f}_R = \hat{f}$, that is $\langle \hat{f}_R \rangle$ converges in the L^2 -sense to \hat{f} .
- (b) Suppose that for any $\delta > 0$, the functions $\langle \hat{f}_R(\xi) \rangle$ converge uniformly to $g(\xi)$ for $\delta \leq |\xi| \leq \delta^{-1}$. Prove that $g = \hat{f}$ almost everywhere.
- (c) Prove that if $f(x) = (x + i\epsilon)^{-1}$ for an $\epsilon > 0$, then

$$\hat{f}(\xi) = -2\pi i \chi_{[0, \infty)}(\xi) e^{-\epsilon\xi}. \quad (5)$$

- (d) Let $f_\epsilon(x) = x^{-1} \chi_{[\epsilon, \infty)}(|x|)$; for $\epsilon > 0$, these functions belong to $L^2(\mathbb{R})$. Compute $\hat{f}_\epsilon(\xi)$. Show that the operators $H_\epsilon g = g \mapsto f_\epsilon * g$ are bounded as maps from L^2 to itself, and

$$\text{s-lim}_{\epsilon \rightarrow 0^+} H_\epsilon = \mathcal{H}. \quad (6)$$

Here \mathcal{H} is the Hilbert transform.

3. Suppose that $\varphi \in L^1(\mathbb{R})$. Show that the integral operator defined for $f \in C_c^0(\mathbb{R})$ by

$$\Phi f(x) = \int_{-\infty}^{\infty} \varphi(x-y) f(y) dy \quad (7)$$

extends to define a bounded map, $\Phi : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$.

4. Let $k(s, t) \in C^0([0, 1] \times [0, 1])$, and, for $f \in C^0([0, 1])$ define

$$Kf(s) = \int_0^s k(s, t) f(t) dt. \quad (8)$$

- (a) Prove that $K : C^0([0, 1]) \rightarrow C^0([0, 1])$ is continuous.
- (b) Let $M = \|k\|_\infty$. Show that, for $s \in [0, 1]$,

$$|Kf(s)| \leq Ms \|f\|_\infty. \quad (9)$$

(c) For each $n \in \mathbb{N}$ show that, for $s \in [0, 1]$,

$$|K^n f(s)| \leq \frac{(Ms)^n}{n!} \|f\|_\infty. \quad (10)$$

(d) Prove that $(\text{Id} - K) : C^0([0, 1]) \rightarrow C^0([0, 1])$ is invertible and the inverse is given by the **norm** convergent series:

$$(\text{Id} - K)^{-1} = \sum_{n=0}^{\infty} K^n. \quad (11)$$

Here the “norm” refers to the operator norm on $\mathcal{L}(C^0([0, 1]), C^0([0, 1]))$.