Math 644, Problem set 1 due September 25, 2007
Dr. Epstein

Reading: Taylor sections 1.1, 1.2, 1.3-5 (for review), 1.6, 1.13, 1.15

1. If f is a C^1-function in an open subset of $\mathbb{R}^2$ containing $[a, b] \times \{y\}$, then prove that

$$\frac{d}{dy} \int_a^b f(x, y) dx = \int_a^b \frac{\partial f}{\partial y}(x, y) dx. \tag{1}$$

2. Suppose that M is a smooth, compact surface in $\mathbb{R}^3$ and that $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field. Assume that if $x \in M$, then $F(x)$ is tangent to M; that is the line $t \mapsto x + tF(x)$ is tangent to M at $t = 0$. Suppose that $x_0 \in M$, show that the solution to the initial value problem

$$\frac{dx}{dt} = F(x) \quad x(0) = x_0, \tag{2}$$

is a curve lying on M and that this solution is defined for all $t \in \mathbb{R}$.

3. Descrbe the solution to the equation

$$u_x^2 + u_y^2 = 1 \text{ with } u(x, y) = 0 \text{ on } \{y = x^2\}, \tag{3}$$

assuming that $u_y > 0$ along the initial parabola and u_x is smooth. In what set is this solution single valued and smooth? Use a computer to draw the level curves of u, and explain how the solution breaks down.

4. Suppose that $f(a, x)$ is a smooth family of functions. Here $a \in U$, an open subset of $\mathbb{R}^m$, and for each $a \in U$, $f(a, x)$ is defined for $x \in V_a$, an open subset of $\mathbb{R}^n$. An envelope of this family is a function $g(x)$ defined over an open set $W \subset \mathbb{R}^n$, such that, for each $x \in W$, there is an $a(x) \in U$ such that

   - $x \in V_{a(x)}$
   - $g(x) = f(a(x), x)$
   - The graph of g at $(x, g(x))$ is tangent to the graph of $f(a(x), \cdot)$ at this point.

In general g need not be smooth everywhere. If it is then we say that g is a smooth envelope. The envelope is non-trivial if $a(x)$ is not constant.

(a) Suppose that the envelope exists and is non-trivial, find algebraic (that is non-differential) equations one could solve to find $a(x)$.

(b) If $m = n$, then find a sufficient condition on $f(a, x)$ for there to be a smooth envelope in a neighborhood of x_0, with $a(x_0) = a_0$. Hint: Look at section 1.3 in Taylor.

(c) Let $f(a, x) = \sqrt{a^2 - (x - 2a)^2}$; for $x \in (a, 3a)$. Find the envelope of this family.

(d) Suppose that $u(a, x)$ is a smooth family of functions such that for each a, $u(a, \cdot)$ solves the first order PDE: $F(x, u, d_x u) = 0$. Show that if v is a smooth envelope of this family, then v also satisfies this PDE.

(e) Let $u(a, x) = a \cdot x$, for $x \in \mathbb{R}^n$, and $a \in \mathbb{R}^n$ of length 1. Each member of this family satisfies $\|d_x u\|^2 = 1$. Find the 2 envelopes of this family, and show, by direct computation, that they solve the PDE in $\mathbb{R}^n \setminus \{0\}$. Hint: Think geometrically!

5. Let $a(u)$ be a smooth function. We consider a first order PDE of the form

$$\partial_t u(x, t) + a(u(x, t))\partial_x u(x, t) = 0 \quad u(x, 0) = v(x) \in C^1(\mathbb{R}). \tag{4}$$

To solve this initial value problem in a neighborhood of $\{t = 0\}$, we use the method of characteristics: For each x_0 define the vector field $V_{x_0} = \partial_t + a(v(x_0))\partial_x$; if (x, t) lies on the integral curve of V_{x_0}, passing through $(x_0, 0)$, then we set $u(x, t) = v(x_0)$. Describe these integral curves. Show that for each $x_0 \in \mathbb{R}$ this defines a solution to (4) in a set of the form $(x_0 - \epsilon, x_0 + \epsilon) \times (-\delta, \delta)$, for positive ϵ, δ, which may depend on x_0.

By considering the equation

$$u_t + uu_x = 0 \text{ with } u(x, 0) = e^{-x^2}, \tag{5}$$

show that the solution may fail to exist globally because there exist pairs $x_1 \neq x_2$, such that $e^{-x_1^2} \neq e^{-x_2^2}$, and the trajectories of V_{x_1} and V_{x_2} through $(x_1, 0)$, $(x_2, 0)$, respectively, cross. Find a condition on $v(x)$ so that the solution to $u_t + uu_x = 0$ is well defined and smooth in $t \geq 0$.