Chapter 3

The complete metric approach to the $\overline{\partial}$–problem

Version: 0.1; Revised:3-17-91; Run: February 5, 1998

3.0 Introduction

The remainder of the course will be concerned with a new technique for solving the $\overline{\partial}$–Neumann problem developed by R. Melrose and myself. Before discussing it, let's consider where this problem came from and what its solution is good for. Recall that a fundamental problem in several complex variables was to prove the Levi conjecture that a pseudoconvex domain is always a domain of holomorphy. Although we did not prove it, we remarked that any pseudoconvex domain can be exhausted by smooth strictly pseudoconvex domains. It is a classical result of Behnke and Stein that a domain in $\mathbb{C}^n$ which is exhausted by domains of holomorphy is itself a domain of holomorphy. Thus putting together these observations we see that to prove the Levi conjecture for domains in $\mathbb{C}^n$ it suffices to prove it for smooth, strictly pseudoconvex domains.

It is possible to prove the Levi conjecture if you can solve the $\overline{\partial}$–equation

$$\overline{\partial} u = \alpha$$

with control on the regularity of u at the boundary. This means that if α is smooth up to the boundary of the domain then so is u. Suppose that Ω is a strictly pseudoconvex domain and $p \in \partial\Omega$. There is an open neighborhood U_p and a function f_p such that

$$f_p \in H(U_p)$$
$$f_p(p) = 0, \operatorname{Re} f_p < 0 \text{ in } \Omega \cap U_p.$$

Choose a function $\psi \in \mathcal{C}_c^\infty(U_p)$ that equals 1 near to p and set $g_p = \psi/f_p$. The $0,1$–form $\alpha_p = \overline{\partial} g_p$, extended to 0 in U_p^c, is clearly smooth and $\overline{\partial}$–closed. Thus if we can find a solution u_p to $\overline{\partial} u_p = \alpha_p$ which is smooth up to $\partial\Omega$ then the function $v_p = f_p - u_p \in H(\Omega)$ blows up exactly as $z \to p$. It follows from Proposition (2.7.13) that Ω is a domain of holomorphy. Of course all that is really required is that u_p is bounded in $\overline{\Omega}$.

If Ω is strictly pseudoconvex with a smooth boundary then it turns out to be possible to carry this program through. This was first accomplished by J.J. Kohn using the so called $\overline{\partial}$–Neumann problem. He considered a second order operator cooked up out of $\overline{\partial}$ by choosing a metric on Ω smooth up to the boundary.

Let $\overline{\partial}^*$ denote the formal adjoint of $\overline{\partial}$ relative to this metric. We can define a quadratic form on smooth p,q–forms by

$$Q(f,f) = \int_{\Omega} [< \overline{\partial} f, \overline{\partial} f > + < \overline{\partial}^* f, \overline{\partial}^* f >] \, \mathrm{dVol}\,. \tag{3.0.1}$$

Using the Friedrichs' extension this in turn defines a self adjoint, second order operator on $L^2(\Omega;\Lambda^{p,q})$. Formally it is given by

$$\Box f = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^* \overline{\partial}) f.$$

The domain of the operator is defined as those forms $f \in L^2$ with a certain amount of additional regularity which satisfy

$$< f, \Box \phi > = Q(f,\phi), \text{ for all smooth } p,q\text{–forms } \phi. \tag{3.0.2}$$

For smooth functions this condition can be reinterpreted as local boundary conditions:

$$\imath^*[N \lrcorner \overline{\partial} f] = 0, \quad \imath^*[\overline{\partial}^* f] = 0, \tag{3.0.3}$$

where $\imath : \partial\Omega \longrightarrow \Omega$ and N is the unit normal to $\partial\Omega$. This is called the $\overline{\partial}$–Neumann boundary condition. To solve the Levi problem one argues as follows: Let α be a closed $0,1$–form, and set $v = \overline{\partial}^* \Box^{-1}\alpha$. One can show that $\overline{\partial}\Box^{-1} = \Box^{-1}\overline{\partial}$ and therefore

$$\overline{\partial} v = \overline{\partial}\,\overline{\partial}^* \Box^{-1}\alpha = \Box\Box^{-1}\alpha = \alpha. \tag{3.0.4}$$

The regularity of v follows from a regularity theory developed for equations of this type by Kohn, et. al.

Note that on $0,0$–forms the boundary condition reduces to $\imath^*[N \lrcorner \overline{\partial} f] = 0$ which is satisfied by any holomorphic function. Thus a smooth function in $H(\Omega)$ is in the domain of $\Box$. From this it is clear that this operator is not elliptic, as it has an infinite dimensional kernel. If one uses the standard euclidean metric to define the adjoints then $\Box = \Delta$ as formal differential operators, so the difficulty lies in the boundary conditions. This also implies that the solution to $\overline{\partial} u = \alpha$ is not unique, even within the domain of $\Box$. However the solution defined above has a special property. Suppose that $h \in \ker \overline{\partial}$ then

$$< v, h > = < \overline{\partial}^* \Box^{-1}\alpha, h > = < \Box^{-1}\alpha, \overline{\partial} h > = 0.$$

In other words v is orthogonal to all L^2–holomorphic functions in Ω.

The difficulty in this method lies in the fact that the differential operator Δ is unrelated to the complex geometry of the Ω whereas the boundary condition is intimately tied to it. As we shall see, this can be rephrased by saying that there are two incompatible homogeneities at the boundary. To replace the boundary condition, (3.0.3), Melrose and I consider a complete metric on Ω and use it to define adjoints and thereby a $\Box$ operator. This has the advantage that there is only one homogeneity, that defined by the complex geometry of the boundary.

To solve the $\overline{\partial}$–Neumann problem we need to replace consideration of $0,0$ and $0,1$–forms by $n,0$ and $n,1$ forms. For a domain in $\mathbb{C}^n$ this poses no problems for if f is a function then setting

$$\omega = dz_1 \wedge \cdots \wedge dz_n$$

we obtain that

$$\overline{\partial}(f\omega) = (\overline{\partial} f) \wedge \omega \tag{3.0.5}$$

and if α is a $0,1$–form then

$$\overline{\partial}(\alpha \wedge \omega) = (\overline{\partial}\alpha) \wedge \omega. \tag{3.0.6}$$

Thus we can solve

$$\overline{\partial}u = \alpha$$

for $\overline{\partial}$–closed $0,1$–forms if and only if we can solve it for $\overline{\partial}$–closed $n,1$–forms.

There is something a bit special about $n,0$–forms, they have a canonical L^2 norm:

$$\|f\|^2 = \int_{\Omega} f \wedge \overline{f}. \tag{3.0.7}$$

This canonical norm agrees with the L^2–norm defined by any choice of hermitian metric in Ω. Thus we see that if u is a smooth function in Ω then $u\omega$ is an square integrable $n,0$–form relative to any hermitian metric on Ω. This observation will allow us to use a complete metric to solve the $\overline{\partial}$–Neumann problem

Before we consider these matters in detail, I would like to quickly review how the calculus of pseudodifferential operators is used invert a elliptic operator on a compact manifold. In the process we will extract the essence of the method which we will subsequently adapt to analyze operators like $\Box$ on strictly pseudoconvex domains with metrics of the form

$$g = -\partial_{z_i}\partial_{\bar{z}_j} \log \rho \, dz_i d\bar{z}_j. \tag{3.0.8}$$

Here ρ is a smooth plurisubharmonic defining function for Ω.

Suppose that M is a compact riemannian manifold with metric g. We denote the smooth functions on M by $\mathcal{C}^\infty(M)$ and its dual space by $\mathcal{C}^{-\infty}(M)$. This is the space of distributions defined on M. Suppose that A is a linear operator

$$A : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^{-\infty}(M).$$

An operator of this type has a kernel belonging to $\mathcal{C}^{-\infty}(M \times M)$. This is called the Schwarz kernel of A, we will denote it by κ_A.

A is a differential operator if its Schwarz kernel is supported along the diagonal in $M \times M$. It is of order m if for any smooth function ψ supported in a coordinate patch we have an expression of the form

$$A(\psi u)(x) = \sum_{\alpha, |\alpha| \leq m} a_\alpha(x) \partial_x^\alpha u(x). \tag{3.0.9}$$

An operator is a pseudodifferential operator if the wave front set of the Schwarz kernel is contained in the conormal bundle of the diagonal in $M \times M$. In simpler terms this means that if we introduce coordinates into disjoint open sets U, V and choose functions $\phi \in \mathcal{C}_c^\infty(U), \psi \in \mathcal{C}^\infty(V)$, then the Schwarz kernel of the operator $u \longrightarrow \phi A \psi u$, is a smooth function on $M \times M$. Whereas the Schwarz kernel of the operator $u \longrightarrow \psi A \psi u$ is given by

$$\kappa_{\psi A \psi} = \int e^{\xi \cdot (x-y)} a(x, \xi) d\xi,$$

where the integral is interpreted to mean

$$\psi A \psi u(x) = \int e^{\xi \cdot x} a(x, \xi) \widehat{u}(\xi) d\xi. \tag{3.0.10}$$

The function $a(x, \xi)$ is called the symbol of A and it satisfies certain estimates: for non–negative multiindices α, β, there are constants $C_{\alpha\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m - |\beta|}. \tag{3.0.11}$$

This is actually too general a class of operators for our purposes; we place the further restriction that the symbol have an expansion for large ξ

$$a(x, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi), \tag{3.0.12}$$

where $a_{m-j}(x,\xi)$ is homogeneous of order $m-j$ is ξ. An operator which satisfies (3.0.10)–(3.0.12) is called a Kohn–Nirenberg pseudodifferential operator of order m. We will denote the set of such operators by $\Psi^m_{\mathrm{KN}}(M)$.

The highest order term $a_m(x,\xi)$ is called the principal symbol of the operator A. It is actually not a function on $M \times M$ but is well defined as a function on T^*M. The variable ξ appearing in (3.0.10) is the dual variable for the fiber of the cotangent bundle defined by the coordinate x. Simply put, it is the trivialization of the cotangent bundle which arises by expressing one forms relative to the basis $dx_1, \dots, dx_n$. For a differential operator as in (3.0.9) the principal symbol is given by

$$\sigma_m(A)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha.$$

Thus a differential operator of order m with smooth coefficients belongs to $\Psi^m_{\mathrm{KN}}(M)$.

Conversely, given a smooth function on T^*M, homogeneous of degree m outside a compact subset, we can define a pseudodifferential operator. Let a be such a function and let U_i be a covering of M by coordinate balls, with coordinates x_i. Choose a partition of unity subordinate to this cover, $\{\psi_i\}$. If ξ_i denote the dual variables then the operator defined by

$$Au = \sum_i \int e^{i\xi_i \cdot x_i} \psi_i(x_i) a(x_i,\xi_i)\widehat{(\psi_i u)}(\xi_i) d\xi_i, \tag{3.0.13}$$

is an element of $\Psi^m_{\mathrm{KN}}(M)$ with principal symbol a.

The correspondence between symbols and operators is bijective in a filtered sense. If $S^m(M)$ denotes functions on $T^*(M)$ homogeneous of degree m satisfying (3.0.11) then there is a one to one mapping

$$S^{\{m\}}(M) = S^m(M)/S^{m-1}(M) \longleftrightarrow \Psi^m_{\mathrm{KN}}(M)/\Psi^{m-1}_{\mathrm{KN}}(M). \tag{3.0.14}$$

We denote the principal symbol by $\sigma_m(A)$; it takes values is $S^{\{m\}}(M)$. A fundamental fact which makes pseudodifferential operators useful is that if $A \in \Psi^m_{\mathrm{KN}}(M), B \in \Psi^n_{\mathrm{KN}}(M)$ then $A \circ B \in \Psi^{m+n}_{\mathrm{KN}}(M)$. Equally important is that

$$\sigma_{m+n}(A \circ B) = \sigma_m(A) \cdot \sigma_n(B). \tag{3.0.15}$$

The pseudodifferential operators form an algebra and the symbol map is an algebra homomorphism.

The final fact of importance is that a pseudodifferential operator of order zero defines a bounded map from $L^2(M)$ to itself. We can define L^2–Sobolev spaces as follows: for each real number s

$$H^s(M) = \{u \in \mathcal{C}^{-\infty}(M);\ Au \in L^2 \text{ for all } A \in \Psi^s_{\mathrm{KN}}(M)\}. \tag{3.0.16}$$

For positive integral values of s it is easy to put an inner product on H^s making it into a complete Hilbert space. Using interpolation this can be extended to all positive real numbers and by duality to negative real numbers. The following theorem follows easily from the definitions and the L^2 result

Theorem 3.0.17. *For every real s an operator $A \in \Psi^m_{\mathrm{KN}}(M)$ defines bounded mappings*

$$A : H^s(M) \longrightarrow H^{s-m}(M). \tag{3.0.18}$$

An opertor is negligible, as a pseudodifferential operator, if it has order $-\infty$. This does not mean that it is the zero operator but only that its principal symbol is zero and its Schwarz kernel belongs to $\mathcal{C}^\infty(M \times M)$. From this we conclude that if $A \in \Psi^{-\infty}_{\mathrm{KN}}(M)$ then

$$A : \mathcal{C}^{-\infty}(M) \longrightarrow \mathcal{C}^\infty(M). \tag{3.0.19}$$

Now that we have assembled all the pieces, we apply these ideas to invert the Laplace operator $\Delta \in \Psi^2_{\mathrm{KN}}(M)$. This operator is not actually invertible as the constant functions lie in its kernel, so instead we consider $\Delta + 1$. The principal symbol of Δ is given by

$$\sigma_2(\Delta) = |\xi|^2, \tag{3.0.20}$$

where $|.|$ is the norm on T^*M defined by the metric. Note that it does not vanish outside the zero section, any pseudodifferential operator with this property is called elliptic.

We wish to find an operator $R(1)$ such that

$$(\Delta + 1)R(1) = \mathrm{Id}\,. \tag{3.0.21}$$

We suppose that there is a pseudodifferential operator that satisfies this operator equation. Since $\sigma_0(\mathrm{Id}) = 1$, (3.0.21) implies that

$$\sigma_2(\Delta + 1)\sigma_{-2}(R(1)) = 1. \tag{3.0.22}$$

We iteratively construct an operator $\widetilde{R}$ so that

$$(\Delta + 1)\widetilde{R} - \mathrm{Id} \sim 0 \tag{3.0.22'}$$

in the sense that this is an operator of order $-\infty$. The first step is to let P_1 be an operator whose principal symbol is $(|\xi|^2 + 1)^{-1}$. This operator is of order -2 using (3.0.15) we compute that

$$E_1 = (\Delta + 1)P_1 - \mathrm{Id} \tag{3.0.23}$$

is in $\Psi^{-1}_{\mathrm{KN}}(M)$, let e_1 denote its symbol. Next we choose an operator P_2 with symbol $-e_1(|\xi|^2 + 1)^{-1}$, again applying (3.0.15) we obtain

$$E_2 = (\Delta + 1)(P_1 + P_2) - \mathrm{Id} \in \Psi^{-2}_{\mathrm{KN}}(M).$$

Clearly we can apply this recursively obtaining operators

$$P_j \in \Psi^{-j-1}_{\mathrm{KN}}(M)$$

such that

$$E_j = (\Delta + 1)(P_1 + \cdots + P_j) \in \Psi^{-j}_{\mathrm{KN}}(M).$$

We want to let

$$\widetilde{R} \sim \sum_{j=1}^{\infty} P_j$$

so that

$$E = (\Delta + 1)\widetilde{R} \in \Psi^{-\infty}_{\mathrm{KN}}(M). \tag{3.0.24}$$

This in fact can be done, using the Borel summation lemma one can show that there is a pseudodifferential operator $\widetilde{R}$ with the property that

$$\widetilde{R} - (P_1 + \cdots + P_j) \in \Psi^{-j-2}_{\mathrm{KN}}(M),\ \forall\, j > 0.$$

We have therefore obtained an operator $\widetilde{R} \in \Psi^{-2}_{\mathrm{KN}}(M)$ satisfying (3.0.22'). What remains is to modify $\widetilde{R}$ so as to obtain the true inverse.

We rewrite (3.0.24) as

$$(\Delta + 1)\widetilde{R} = \mathrm{Id} + E$$

where E has a smooth kernel. If $\mathrm{Id} + E$ is invertible then one can very easily show that $(\mathrm{Id} + E)^{-1} = \mathrm{Id} + F$ where F also has a smooth kernel. Thus setting

$$R(1) = \widetilde{R}(\mathrm{Id} + F)$$

we obtain an operator in $\Psi^{-2}_{\mathrm{KN}}(M)$ which inverts $\Delta + 1$.

It may happen that $\mathrm{Id} + E$ is not invertible, let $u_1, \ldots, u_l$ denote a basis for the kernel. Since E has a smooth kernel it is immediate that

$$u_i \in \mathcal{C}^\infty(M), i = 1 \ldots, l.$$

If we take the adjoint of the equation satisfied by $\widetilde{R}$ we obtain

$$\widetilde{R}^*(\varDelta + 1) = \mathrm{Id} + E^* \tag{3.0.25}$$

We know, a priori, that $(\varDelta + 1)^{-1}$ exists, at least on L^2 thus we can find functions $v_i \in L^2$ such that

$$(\varDelta + 1)v_i = u_i.$$

Using (3.0.25) we can show that $v_i \in \mathcal{C}^\infty(M)$ as well. If we replace $\widetilde{R}$ with the kernel

$$\widetilde{R}' = \widetilde{R} + \sum_{i=1}^{l} v_i \otimes u_i$$

then

$$(\varDelta + 1)\widetilde{R}' = \mathrm{Id} + E'$$

where $(\mathrm{Id} + E')$ is invertible. Since the correction has a smooth kernel we obtain as before that $R(1) \in \Psi^{-2}_{\mathrm{KN}}(M)$. A similar analysis applies to show that the inverse of any invertible elliptic pseudodifferential operator of order m is an elliptic pseudodifferential operator of order $-m$.

We can also obtain a smoothing error term by using the composition properties of the calculus directly. We return to (3.0.23) which we rewrite as

$$(\varDelta + 1)P_1 = \mathrm{Id} + E_1. \tag{3.0.23'}$$

Since $E_1 \in \Psi^{-1}_{\mathrm{KN}}$ it follows from (3.0.15) that

$$E_1^k \in \Psi^{-k}_{\mathrm{KN}}. \tag{3.0.26}$$

Using a summation argument similar to that used above to construct $\widetilde{R}$ we can construct an operator $\widetilde{F} \in \Psi^{-1}_{\mathrm{KN}}$ such that

$$\mathrm{Id} + \widetilde{F} \sim \sum_{j=0}^{\infty} (-E_1)^j. \tag{3.0.27}$$

That the series is summable follows from (3.0.26). As a consequence of (3.0.27) we obtain that

$$(\mathrm{Id} + E_1)(\mathrm{Id} + \widetilde{F}) - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{KN}}. \tag{3.0.28}$$

Thus we can replace the iteration step above with this conceptually simpler construction. If we let $\widehat{R} = P_1(\mathrm{Id} + \widetilde{F})$ then

$$(\varDelta + 1)\widehat{R} - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{KN}}.$$

To finish let's assemble the properties of pseudodifferential operators which allowed us to invert the Laplacian.

(3.0.29) Pseudodifferential operators form an algebra which is closed under formal adjoints.

(3.0.30) There is a symbol map which is an algebra homomorphism and bijective, in a filtered sense.

(3.0.31) Pseudodifferential operators define bounded maps between a scale of Hilbert spaces H^s which satisfy

$$\mathcal{C}^\infty(M) = \bigcap_s H^s, \quad \mathcal{C}^{-\infty}(M) = \bigcup_s H^s.$$

(3.0.32) The residual operators (those of order $-\infty$) form an ideal.

We will soon see how these principles guide the construction of an operator algebra which leads to the inversion of the Laplace operator defined by a complete metric on a strictly pseudoconvex domain. What we need to add to these general principles is a means for handling boundary conditions. To that end we need to consider a special case in detail. The thrust of the method is to then reduce the general case to the special case.

3.1 The unit ball

The unit disk is a model for the complete, simply connected Riemannian manifold with constant negative curvature. The metric can be written in the form

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}. \tag{3.1.1}$$

We can identify the tangent space to the unit disk with the vectors of type $1,0$. When represented relative to the basis $\partial_z, \partial_{\bar{z}}$ a real vector takes the form

$$X = \alpha\partial_z + \overline{\alpha}\partial_{\bar{z}}. \tag{3.1.2}$$

From (3.1.2) it is clear that the map

$$X \longrightarrow \alpha\partial_z = Z_X$$

defines an isomorphism of $T\mathbb{CB}^1$ with $T^{1,0}\mathbb{CB}^1$ as a real vector space.

The $(1,1)$–form,

$$\omega = -\partial\overline{\partial}\log(1-|z|^2)$$

defines a hermitian pairing on $T^{1,0}$ by

$$h(W,Z) = \omega(W,\overline{Z}).$$

A simple calculation shows that

$$ds^2(X,Y) = 4\,\mathrm{Re}\,h(Z_X,Z_Y).$$

Thus we have a relation between the strictly plurisubharmonic defining function and the hyperbolic geometry of the unit disk.

Using this connection we can easily show that the metric is invariant under all biholomorphic self maps of the unit disk. The Schwarz lemma implies that all such mappings are of the form:

$$w = \gamma z = \frac{az+b}{\overline{b}z+\overline{a}},\ |a|^2-|b|^2 = 1.$$

An elementary computation establishes that

$$\gamma^*(1-|z|^2) = \frac{(1-|z|^2)}{(\overline{b}z+\overline{a})(b\bar{z}+a)}. \tag{3.1.3}$$

Because γ is a holomorphic map,

$$\gamma^*\partial\overline{\partial}\log(1-|z|^2) = \partial\overline{\partial}\gamma^*\log(1-|z|^2) = \partial\overline{\partial}\log(1-|z|^2), \tag{3.1.4}$$

from which the claim follows. Thus the biholomorphic self maps are isometries relative to the constant curvature metric and $\mathbb{CB}^1$ is a homogeneous space. It is isomorphic to

$$\mathbb{CB}^1 = SU(1,1)/U(1). \tag{3.1.5}$$

A simple count of dimensions shows that all orientation preserving isometries are of this form. Note that the element $-\mathrm{Id}$ acts trivially on the disk, so the true automorphism group is $PSU(1,1) = SU(1,1)/\{\mathrm{Id},-\mathrm{Id}\}$.

For the purposes of generalization there is a different model which is better suited to computation. To define the projective model we consider $\mathbb{C}^2$ with the hermitian inner product

$$< X, Y >= X_1\overline{Y}_1 - X_2\overline{Y}_2.$$

A simple calculation verifies that if $A \in SU(1,1)$ then $< AX, AY >=< X, Y >$. Of course the metric induced by $\mathrm{Re} < \cdot,\cdot >$ has signature $(2,2)$ however if we restrict to the hypersurface given by

$$H = \{X;\ < X, X >= -1\}$$

then we get a metric of signature $(2,1)$. The unit circle acts on H via $X \longrightarrow e^{i\theta}X$. This action commutes with the action of $SU(1,1)$ on H and therefore allows us to define a metric on the quotient $H/U(1)$ by identifying the tangent space of the quotient with the orthocomplement of the vector field which generates the $U(1)$ action. Since the $U(1)$–action commutes with the action of $SU(1,1)$ this group acts as isometries of the quotient space. It is sometimes useful to have a second representation, we set

$$N = \{X :< X, X >< 0\} \text{ then } H/U(1) = N/\mathbb{C}^*.$$

If we use z_1, z_2 as local coordinates for $\mathbb{C}^2$ then the projection

$$\pi(z_1, z_2) = \frac{z_1}{z_2}$$

carries N onto $\mathbb{CB}^1$. In this representation it is a simple matter to deduce (3.1.3) and therefore (3.1.4). Suppose that $A \in \mathrm{SU}(1,1)$ then the action of A on $\mathbb{CB}^1$ is defined as follows,

$$A \cdot z = \frac{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_1}{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_2}. \tag{3.1.6}$$

By definition $A^*(|z_1|^2 - |z_2|^2) = |z_1|^2 - |z_2|^2$ thus setting $\rho = |z|^2 - 1$ we have

$$A^*(\rho) = \frac{\rho}{|(A\begin{pmatrix} z \\ 1 \end{pmatrix})_2|^2}. \tag{3.1.7}$$

The important point being that $A^*\rho/\rho$ is the squared modulus of a holomorphic function.

This construction easily generalizes to n–dimensions. Our goal is to find a metric on the unit ball in $\mathbb{CB}^n$ which is invariant under a very large group of holomorphic transformations. As before we consider $\mathbb{C}^{n+1}$ with the lorentz metric

$$< X, Y >= X_1\overline{Y}_1 + \cdots + X_n\overline{Y}_n - X_0\overline{Y_0}.$$

The lie group $SU(n,1)$ is defined as those matrices of determinant 1 such that

$$< AX, AY >=< X, Y >, \text{ for all } X, Y \in \mathbb{C}^{n+1}.$$

As before we let

$$H = \{X \in \mathbb{C}^{n+1}; < X, X >= -1\}, \quad N = \{X \in \mathbb{C}^{n+1}; < X, X >< 0\}.$$

Evidently H is invariant under the action of $SU(n,1)$ and also the action of $U(1)$ defined by $X \longrightarrow e^{i\theta}X$.

We define a projection of N into $\mathbb{C}^n$ by

$$\pi(X) = (\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}).$$

One easily sees that the image of H is exactly the interior of the unit ball. The map is not one to one as X and $e^{i\theta}X$ have the same projection, but this is easily seen to be the only possibility. Therefore

$$\mathbb{CB}^n \simeq H/U(1).$$

To obtain a representation as a homogeneous space we observe that

$$H/U(1) \simeq SU(n,1)/U(n). \tag{3.1.8}$$

Since $SU(n,1)$ acts transitively on $H/U(1)$ to prove (3.1.8) we need only compute the stabilizer of a single fiber in. This is simplest for the equivalence class $[(0,\ldots,0,e^{i\theta})]$ The stabilizer is easily seen to have the form

$$\begin{bmatrix} & & & 0 \\ & e^{-i\theta}SU(n) & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & e^{in\theta} \end{bmatrix}.$$

This is simply $U(n) \longrightarrow SU(n,1)$ as asserted.

The action of $SU(n,1)$ on $\mathbb{CB}^n$ is defined as before by

$$A\cdot z = \pi(A\pi^{-1}z) = \left(\frac{(A\begin{pmatrix} z \\ 1\end{pmatrix})_1}{(A\begin{pmatrix} z \\ 1\end{pmatrix})_0}, \ldots, \frac{(A\begin{pmatrix} z \\ 1\end{pmatrix})_n}{(A\begin{pmatrix} z \\ 1\end{pmatrix})_0} \right).$$

Using this formula and the invariance of the lorentz inner product one easily derives that

$$A^*(\rho) = \frac{\rho}{|(A\begin{pmatrix} z \\ 1\end{pmatrix})_0|^2}. \tag{3.1.9}$$

As before the ratio $A^*\rho/\rho$ is the squared modulus of a holomorphic function.

Therefore the invariant metric on $\mathbb{CB}^n$ is given, as before, by

$$g = -\partial\overline{\partial}\log\rho$$

To see that it is invariant we need to show that $A^*g = g$ for all A in $SU(n,1)$. This follows immediately from (3.1.9) because the denominator is the squared modulus of a holomorphic function. This metric is called the Bergman metric. One can show that it agrees with the metric induced by π from the inner product $< \cdot, \cdot >$ on H, restricted to the orthogonal trajectories of the action of $U(1)$.

Using the group invariance one can show that if z, w are two points in $\mathbb{CB}^n$ then

$$\cosh\frac{1}{2}d(z,w) = \frac{|1-(z,w)|}{[(1-|z|^2)(1-|w|^2)]^{\frac{1}{2}}}. \tag{3.1.10}$$

Here $(\cdot,\cdot)$ is the standard hermitian inner product on $\mathbb{C}^n$.

Exercise 3.1.11**.**

(1) Prove that the function defined on the right hand side of (3.1.10) is invariant under the action of $SU(n,1)$.
(2) Prove that the real lines through 0 are geodesics of the Bergman metric.
(3) Prove the formula (3.1.10), hint: show that it suffices to consider $z = 0, w = (\zeta, 0, \ldots, 0)$ and then compare (3.1.10) with the result of integrating the Bergman metric.

The group $SU(n,1)$ therefore acts as isometries of $\mathbb{CB}^n$ with the Bergman metric, however the action is not effective. The center of the group

$$Z_n = \{\begin{bmatrix} e^{im\omega}\mathrm{Id}_n & 0 \\ 0 & e^{im\omega}\end{bmatrix}, m\in\mathbb{Z}, \omega = \frac{2\pi}{n+1}\},$$

acts trivially. A dimension count shows that all biholomorphic isometries of the Bergman metric arise from elements of $SU(n,1)$. One can also show that all orientation preserving isometries of the Bergman metric are necessarily biholomorphic maps, thus

$$\mathrm{Isom}(\mathbb{CB}^n) = SU(n,1)/Z_n.$$

One can even prove a sort of converse: that every biholomorphic self map of $\mathbb{CB}^n$ is an isometry of the Bergman metric. This uses a generalization of the Schwarz Lemma. From this we conclude that the group of biholomorphic self maps of the unit ball is isomorphic to $SU(n,1)/Z_n$.

Exercise 3.1.12**.**

(1) Prove that any biholomorphic self map of the unit ball defines an isometry of the Bergman metric. hint: use (2.2.17) and composition with elements of $SU(n,1)$.

(2) Prove that any isometry of the Bergman metric is a biholomorphic mapping. hint: By composing with elements of $SU(n,1)/Z_n$ one can reduce consideration to an isometry fixing the origin and consider the tangent map only at 0.

(3) Prove that the volume form of the Bergman metric is given by

$$\mathrm{dVol} = \frac{c_n \, \mathrm{dV}_{\mathrm{euclid}}}{(1-|z|^2)^{n+1}}, \tag{3.1.13}$$

where c_n is a dimensional constant.

3.2 Analysis on the unit ball

In this section we consider two analytic problems on the unit ball in $\mathbb{C}^n$: the first is constructing the resolvent kernel for the Laplace operator, the second is the construction of the Bergman projector. To keep technical difficulties to a minimum we only construct the resolvent for the Laplace operator acting on functions. Similar considerations lead to an analogous construction for the Laplace operator on p,q–forms.

As remarked in the introduction the space of $n,0$–forms has a canonical bilinear pairing which induces an L^2–structure. Bounded holomorphic $n,0$–forms belong to $L^2(\mathbb{CB}^n;\Lambda^{n,0})$, we denote the space of all L^2 holomorphic $n,0$ forms by $\mathcal{H}^2(\mathbb{CB}^n)$. The orthogonal projection from $L^2(\mathbb{CB}^n;\Lambda^{n,0})$ onto $\mathcal{H}^2(\mathbb{CB}^n)$ is called the Bergman projector. It can be represented by a kernel of the form

$$B(z,w)dz\wedge d\bar{w};$$

the action is then given by

$$\mathcal{B}\omega(z) = \int\limits_{\mathbb{CB}^n} \omega\wedge B(z,w)dz\wedge d\bar{w}.$$

Much of the analysis of holomorphic functions in several variables can be reduced to an analysis of the Bergman kernel function. Once one has solved the $\overline{\partial}$–Neumann problem one can easily construct the Bergman kernel. On the unit ball it is easy to construct the Bergman kernel directly. We conclude this section with that computation as motivation for what comes in the later sections.

To study the resolvent kernel for the Laplacian we employ the group invariance of the Laplace operator. Recall that the Bergman metric on the unit ball is given by

$$g_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{1-|z|^2} + \frac{z_{\bar{\imath}} z_j}{(1-|z|^2)^2}.$$

We can easily show that if $\gamma\in \mathrm{Isom}(()\mathbb{CB}^n)$ and $f\in\mathcal{C}^\infty_c(\mathbb{CB}^n)$ then

$$\gamma^*(\Delta_{\mathrm{B}} f) = \Delta_{\mathrm{B}}\gamma^* f. \tag{3.2.1}$$

The resolvent kernel is defined by the distributional equation

$$(\Delta_{\mathrm{B}}-\lambda)R(\lambda) = \delta_\Delta \tag{3.2.2}$$

and decay properties for $\lambda\notin \operatorname{spec}\Delta_{\mathrm{B}}$.

Suppose that we could find a fundamental solution with pole located at $0\in\mathbb{CB}^n$, i.e. a solution to

$$(\Delta_{\mathrm{B}}-\lambda)F_0 = \delta_0, \tag{3.2.3}$$

which is square integrable near to $|z|=1$ if $\lambda\notin\operatorname{spec}\Delta_{\mathrm{B}}$. Let γ carry p to 0 then (3.2.1) implies that

$$F_p(z) = F_0(\gamma\cdot z)$$

satisfies (3.2.3) with the singularity moved to p. Since the resolvent kernel is unique we do not expect the solution we obtain to (3.2.3) to depend on the 'angle', for otherwise we could obtain different functions F_p by choosing different different group elements γ with $\gamma \cdot p = 0$.

In fact if $\gamma \in \mathrm{Isom}(()\mathbb{CB}^n)$ fixes 0 then $\gamma^* F_0$ is another solution to (3.2.3). By averaging over the stabilizer of $0 \simeq U(n)$ we obtain a solution to (3.2.3) which also satisfies

$$\gamma^* F_0 = F_0, \text{ for all } \gamma \in U(n). \tag{3.2.4}$$

It is easy to see that any function which satisfies (3.2.4) is of the form

$$F_0(z) = f(r^2), \text{ with } r^2 = |z|^2. \tag{3.2.5}$$

An elementary calculation shows that with $\tau = r^2$,

$$\Delta_{\mathrm{B}} F_0 = \tau(1-\tau)^2\left(f_{\tau\tau} + \left[\frac{n}{\tau} + \frac{n-1}{1-\tau}\right] f_\tau\right). \tag{3.2.6}$$

Exercise 3.2.7. Prove (3.2.6), hint: for functions in $\mathcal{C}_c^\infty$ the Bergman laplacian is defined by

$$\int_{\mathbb{CB}^n} (\Delta_{\mathrm{B}} f)\overline{g}\, \mathrm{dVol} = \int_{\mathbb{CB}^n} < \overline{\partial} f, \overline{\partial} g > \mathrm{dVol}. \tag{3.2.7}$$

If we reparametrize the eigenvalue by

$$\lambda = s(n-s)$$

then the equation for the radial fundamental solution is a classical P-Riemann equation:

$$\mathcal{P}\left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & 0 & s & ;\tau \\ -n & 0 & n-s & \end{matrix} \right\}. \tag{3.2.8}$$

If you are unfamiliar with this notation I suggest that you consult Modern Analysis by Whittaker and Watson. Briefly it states that the equation has regular singular points at $0, 1, \infty$ with indicial roots $(0, -n)$ at 0, $(0, 0)$ at ∞ and $(s, n-s)$ at 1.

The reason that we introduce the parameter s is so that the indicial roots are both analytic functions of the parameter, this in turn produces a solution which depends analytically on the parameter. The parameter s defines a two fold cover of the energy parameter λ. The half plane $\mathrm{Re}\, s > \frac{1}{2}n$ is the 'physical' half plane, the spectrum corresponds to $\mathrm{Re}\, s = \frac{1}{2}n$. The half plane $\mathrm{Re}\, s < \frac{1}{2}n$ is the non–physical half plane and it has interpretations in term of scattering theory.

In any case the solution of (3.2.8) which we need can be expressed in term of classical functions by

$$r(\tau; s) = c_n \frac{\Gamma(s)^2}{\Gamma(2s-n+1)} (\tau - 1)^s {}_2F_1(s, s; 2s-n+1; 1-\tau). \tag{3.2.9}$$

Using well known facts about these functions we can show that for $\mathrm{Re}\, s > \frac{1}{2}n$, $r(|z|^2; s)$ is square integrable, near to $|z| = 1$, with respect to the Bergman metric. It has a singularity at 0 such that

$$\int_{\mathbb{CB}^n} r(|z|^2; s)(\Delta_{\mathrm{B}} - s(n-s)) f(z)\, \mathrm{dVol} = f(0), \tag{3.2.10}$$

for $f \in \mathcal{C}_c^\infty(\mathbb{CB}^n)$.

The parameter τ has a more invariant interpretation, it is given by

$$\tau = 1 - [\cosh \frac{1}{2} d(z, 0)]^{-2}. \tag{3.2.10}$$

From this it follows easily that

$$R(z, w; s) = r(1 - [\cosh \frac{1}{2} d(z, w)]^{-2}; s). \tag{3.2.11}$$

In (3.1.13) we showed that

$$\cosh \frac{1}{2} d(z, w) = \frac{|1 - (z, w)|}{[(1 - |z|^2)(1 - |w|^2)]^{\frac{1}{2}}}.$$

We set

$$\iota(z, w) = \frac{|1 - (z, w)|}{[(1 - |z|^2)(1 - |w|^2)]^{\frac{1}{2}}},$$

then

$$R(z, w; s) = \iota(z, w)^{-2s} G_n(\iota(z, w)^{-2}; s), \tag{3.2.12}$$

where $G_n(t, s)$ is analytic for $t \in [0, 1)$ and has a pole at $t = 1$. The main conclusion we draw from (3.2.12) is that if we understand the singularities of the function $\iota(z, w)$ on $\mathbb{CB}^n \times \mathbb{CB}^n$ then it is relatively simple matter to understand the singularities of the resolvent kernel. As a function of s, $G_n(\cdot; s)$ is analytic in $\operatorname{Re} s > 0$ with simple poles at $-\mathbb{N}_0$.

To conclude this section we construct the Bergman kernel function. As before we make use of the group invariance. By definition, the Bergman projector acts like the identity on holomorphic $n, 0$–forms. More explicitly, if $f dz$ is holomorphic then

$$f(z)dz = \int\limits_{\mathbb{CB}^n} f(w)dw \wedge B(z, w)d\bar{w} \wedge dz. \tag{3.2.13}$$

Since the components of a holomorphic $n, 0$ form are also harmonic with respect to the euclidean Laplacian it follows from the mean value theorem that

$$f(0)dz = c_n \int\limits_{\mathbb{CB}^n} f(w)dw \wedge d\bar{w} \wedge dz. \tag{3.2.14}$$

Suppose that $\gamma \in \operatorname{Isom}(()\mathbb{CB}^n)$ carries 0 to z then

$$\gamma^{-1\,*}(\gamma^*(f dz)(0)) = f(z)dz. \tag{3.2.15}$$

By combining (3.2.14) and (3.2.15) we obtain

$$\begin{aligned} f(z)dz &= c_n \int\limits_{\mathbb{CB}^n} \gamma^*(f(w)dw) \wedge d\bar{w} \wedge \gamma^{-1\,*} dz \\ &= c_n \int\limits_{\mathbb{CB}^n} f(w)dw \wedge \gamma^{-1\,*}(d\bar{w} \wedge dz). \end{aligned} \tag{3.2.16}$$

To complete the construction all we need is to compute the Jacobian of γ^{-1}. This we leave as an exercise, the answer is

$$\gamma^{-1\,*}(d\bar{w} \wedge dz) = \frac{d\bar{w} \wedge dz}{(1 - (z, w))^{n+1}}, \tag{3.2.17}$$

where

$$(z, w) = \sum_{i=1}^{n} z_i \bar{w}_i.$$

The formula we finally obtain is that

$$\mathcal{B}(f dz) = c_n \int\limits_{\mathbb{CB}^n} \frac{f(w) dw \wedge d\bar{w} \wedge dz}{(1 - (z, w))^{n+1}}. \tag{3.2.18}$$

How do we know this is the correct kernel? It is uniquely determined by three properties: the range of $\mathcal{B}$ is $\mathcal{H}^2$, it is hermitian symmetric and $\mathcal{B}^2 = \mathcal{B}$. This is equivalent to the statement that it is an *orthogonal* projection onto $\mathcal{H}^2$. The hermitian symmetry is apparent from the formula. Since $B(z, w)$ depends holomorphically on z it follows that $\mathcal{B}f(z)$ is a holomorphic function of z. Furthermore one can adapt the for going argument to show that:

$$B(z, w) = \int\limits_{\mathbb{CB}^n} B(z, x) \wedge B(x, w).$$

Thus we have proved the following theorem:

Theorem 3.2.19**.** *The Bergman projector for the the unit ball in $\mathbb{C}^n$ is given by the following kernel*

$$B(z, w) = \frac{d\bar{w} \wedge dz}{(1 - (z, w))^{n+1}}.$$

Once again we see that understanding the singularities of the Bergman kernel is reduced to studying the singularities of a very simple function $(1 - (z, w))^{-1}$. Clearly this function is smooth away from the intersection of the diagonal in $\mathbb{CB}^n \times \mathbb{CB}^n$ with the boundary. On the other hand it seems to have a very similar sort of a singularity as that which arose in the kernel of the resolvent for the Bergman laplacian. In the next lecture we discuss a method for analyzing such singularities. It amounts essentially to introducing polar coordinates.

3.3 Polyhomogeneous conormal distributions

A large part of the study of pseudodifferential operators on strictly pseudoconvex domains is related to the analysis of singularities. In order to proceed, we must define a class of 'good' singularities. Recall that a function $a(x, \xi)$ is called a symbol of order m if it satisfies the estimates

$$\sup |(1 + |\xi|)^{|\beta| - m} \partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < \infty, \tag{3.3.1}$$

for all non–negative multiindices α, β.

The kernel defined by $a(x, \xi)$ is given by the oscillatory integral

$$K_a(x, x - y) = \int\limits_{\mathbb{R}^n} a(x, \xi) e^{i\xi \cdot (x-y)} d\xi.$$

So as to avoid unimportant technicalities we assume that the order of a is less that $-n$ and therefore the integral exists as an absolutely convergent integral. We would like to investigate the regularity properties of the kernel itself which follow from (3.3.1). For this purpose it is convenient to replace $x - y$ by a new variable z. The x, z coordinates have a nice geometric interpretation, the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ is defined by $z = 0$. Thus we can think of x as a parameter along the diagonal and z as a transverse parameter.

We've assumed that $K_a(x, z)$ belongs to L^∞ it is clear from (3.3.1) that $\partial_x^\alpha K_a$ is also bounded for any multiindex α. On the other hand it is clear that if we take enough derivatives in the z–variables the integral will no longer be absolutely convergent, in fact it will no longer be bounded along the diagonal. However observe that

$$
\begin{aligned}
z_i\partial_{z_j}K_a(x,z) &= \int_{\mathbb{R}^n} a(x,\xi) i z_i \xi_j e^{i\xi\cdot z} d\xi \\
&= \int_{\mathbb{R}^n} a(x,\xi)\xi_j \partial_{\xi_i} e^{i\xi\cdot z} d\xi \\
&= -\int_{\mathbb{R}^n} \partial_{\xi_i}(\xi_j a(x,\xi)) e^{i\xi\cdot z} d\xi
\end{aligned} \tag{3.3.2}
$$

The last equality is obtained by integrating by parts. In virtue of (3.3.1) the symbol $\partial_{\xi_i}(\xi_j a(x,\xi))$ satisfies exactly the same estimates as $a(x,\xi)$ thus we see that the differential operators $z_i\partial_{z_j}$ also preserve the boundedness properties of the kernel.

This seems a bit mysterious, however the span of the differential operators

$$\mathcal{V}(\mathbb{R}^{2n};\Delta) = \mathrm{Span}\{\partial_{x_i}, z_i\partial_{z_j}; i,j = 1,\ldots n\}$$

also has a simple geometric interpretation: these are precisely the smooth vector fields tangent to the diagonal. So we see that the property of the kernel which corresponds to the estimates (3.3.1) is that if $V_1,\ldots,V_l \in \mathcal{V}(\mathbb{R}^{2n};\Delta)$ then

$$V_1\ldots V_l K_a(x,z)$$

has the same regularity as K_a itself. Such a distribution is called a conormal distribution relative to $z=0$.

More generally if X is a manifold and $Y\subset X$ is an embedded submanifold then we can define distributions conormal relative to Y. To do this we use the grading of the distributions on X defined by the L^2–Sobolev spaces. Let $\mathcal{V}(X;Y)$ denote all smooth vector fields on X which are tangent to Y. Let $p\in Y$, we can find coordinates $(y_1,\ldots,y_{n-k},z_1,\ldots,z_k)$ for X such that Y is given by $z_1=\cdots=z_k=0$. In this coordinate patch

$$\mathcal{V}(X;Y) = \mathrm{Span}\{\partial_{y_i}, z_j\partial_{z_l}\}. \tag{3.3.3}$$

We define the conormal distributions along Y with L^2–order s by

$$IH^s(X;Y) = \{u\in H^s(X);\ \text{for any } l, V_1\ldots V_l u\in H^s(X), V_i\in\mathcal{V}(X;Y), i=1,\ldots,l\}.$$

Loosely speaking these are distributions whose regularity is not affected by taking derivatives tangent to Y.

One can show that every such distribution has a representation in a local coordinate patch as an oscillatory integral:

$$u(y,z) = \int_{\mathbb{R}^k} a(y,\eta)e^{i\eta\cdot z}d\eta. \tag{3.3.4}$$

The function $a(y,\eta)$ is a symbol in $\mathbb{R}^{n-k}\times\mathbb{R}^k$. This means that for all non–negative multiindices α,β

$$\sup|(1+|\eta|)^{|\beta|-m}\partial_y^\alpha\partial_\eta^\beta a(y,\eta)|<\infty. \tag{3.3.5}$$

This is simply a generalization of (3.3.1). We denote the space of such functions by $S^m(\mathbb{R}^{n-k}_y\times\mathbb{R}^k_\eta)$. This representation leads to a symbol mapping for a conormal distribution. Invariantly the symbol takes values in $N^*(Y)$, the conormal bundle of $Y\longrightarrow X$.

There is an inequality which pertains between s and m. Instead of exploring this direction it is simpler to redefine the grading on the family of conormal distributions. We define $I^m(X;Y)$ as those distributions on X whose singular support is contained in Y and such that in every local coordinate system as above u has a representation as in (3.3.4) where $a\in S^{m-\frac{1}{2}n+\frac{1}{4}4}(\mathbb{R}^{n-k}_y\times\mathbb{R}^k_\eta)$. This peculiar normalization of the order is useful in other applications and is relatively standardized. The space conormal distributions relative to Y is

$$I(X;Y) = \bigcup_{m\in\mathbb{R}} I^m(X;Y) = \bigcup_{s\in\mathbb{R}} IH^s(X;Y).$$

It is useful to have the two different grading because they are related and it is sometimes easier to prove things about the one than the other.

Recall that we actually required more about of the symbols that defined our pseudodifferential operators than the estimates (3.3.1). We also insisted that they should have an asymptotic expansion into homogeneous terms. If a conormal distribution has a symbol with such an expansion then it will also have an expansion. We consider a simple special case. Suppose that $a(\xi) \in S^m(\mathbb{R}^k)$.

We suppose further that $a(\xi)$ is homogeneous of degree m for $|\xi| > 1$. We call such a function asymptotically homogeneous. The fourier transform of a defines a distribution

$$u(z) = \int_{\mathbb{R}^k} a(\xi) e^{i\xi\cdot z} d\xi. \tag{3.3.6}$$

If $m < -k$ then this integral converges absolutely, otherwise we define the distribution as an oscillatory integral. If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^k)$ then

$$< u, \psi >= \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^k} e^{z\cdot\xi} \psi(z) dz \right) a(\xi) d\xi.$$

Integrating in this order is easily seen to define a tempered distribution.

Since $a(\xi)$ is asymptotically homogeneous of degree m it follows from the oscillatory definition of u that we can integrate by parts to obtain that:

$$(z \cdot \partial_z + k + m)u = \int_{\mathbb{R}^k} (m - \xi \cdot \partial_\xi) e^{i\xi\cdot z} a(\xi) d\xi. \tag{3.3.7}$$

The derivatives on the left are in the sense of distributions. The integrand in (3.3.7) is smooth and compactly supported. We state this result as a lemma.

Lemma 3.3.8. *If $a(\xi) \in \mathcal{C}^\infty(\mathbb{R}^k)$ and homogeneous of degree m for $|\xi| > 1$ then $u(z)$ defined as an oscillatory integral by* (3.3.6) *satisfies:*

$$v = (z \cdot \partial_z + k + m)u \in \mathcal{C}^\infty(\mathbb{R}^k). \tag{3.3.9}$$

From this we can deduce that u is homogeneous of degree $-(k+m)$ up to a smooth error. To that end we observe that $u(z)$ is represented by a smooth function in the complement of $z = 0$. Furthermore

$$u(z) - t^{m+k} u(tz) = \int_t^1 s^{m+k-1} v(sz) ds.$$

This implies that

$$u(tz) = t^{-(m+k)} u(z) - t^{-(m+k)} \int_t^1 s^{m+k-1} v(sz) ds.$$

We apply Taylors theorem with remainder to obtain that

(3.3.10)

$$\begin{aligned} u(tz) &= t^{-(m+k)} u(z) - \\ &\qquad t^{-(m+k)} \sum_{\alpha;\, |\alpha| \le N} \int_t^1 v_\alpha s^{|\alpha|+m+k-1} z^\alpha ds + \sum_{\beta;\, |\beta| = N+1} \int_t^1 s^{m+k+N} z^\beta h_\beta(sz) ds) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned}$$

Here N is some integer chosen so that $m+k+N>0$. The functions $h_\beta(z)$ are smooth. The first two terms are easily analyzed, we let $z=\omega$ a unit vector. Performing the integral and grouping terms we see that

$$\text{I}+\text{II}=t^{-(m+k)}(u(\omega)-\sum_{|\alpha|\leq N}v'_\alpha\omega^\alpha+g_N(t\omega)\log t)+f_N(t\omega), \tag{3.3.11}$$

where $f_N(z), g_N(z)$ are polynomials and $g_N(z)=O(|z|^{-(m+k)})$.

To complete the analysis we need to control the error term. In virtue of our choice of N this integral converges absolutely as $t\to 0$. We rewrite this term as

$$\begin{aligned}\text{III}=&t^{-(m+k)}\sum_{|\beta|=N+1}\int_0^s s^{m+k+N}\omega^\beta h_\beta(s\omega)ds-\\ &\int\sum_{|\beta|=N+1}\int_0^\sigma \tau^{m+k+N}(t\omega)^\beta h_\beta(\sigma t\omega)d\sigma.\end{aligned} \tag{3.3.12}$$

In the second integral we have set $s=t\tau$. Thus we obtain that

$$\text{III}=t^{-(m+k)}h_N(t\omega)+k_N(t\omega),$$

where $h_N(z), k_N(z)$ are smooth functions.

Putting these calculations together we obtain:

Lemma 3.3.13. *If $a(\xi)\in S^m(\mathbb{R}^k)$, for an $m<-k$, is homogeneous for $|\xi|>1$ then the distribution $u(y)$ defined by* (3.3.6) *has an expansion at zero of the form*

$$u(r\omega)=r^{-(m+k)}(f(\omega)+\log r g(r\omega))+h(r\omega). \tag{3.3.14}$$

Here f is a smooth function on the unit sphere, g, h are smooth in a neighborhood of $0\in\mathbb{R}^k$ and $g(z)=O(|z|^{-(m+k)})$.

This result has a generalization to distributions of any order. One uses Lemma (3.3.8). For values of $m>0$ one must allow for the 'homogeneous' distributions of the form

$$u(y)=\partial_y^\beta\delta_0(y).$$

In general (3.3.14) is replaced by a sum of such terms and homogeneous functions of the more familiar kind. Note that this is needed in order for the distribution to be defined as a principal value. On the other hand no log–term arises in this case. We leave the details as a highly recommended exercise. Note that we could rephrase (3.3.13) for any order m by considering u as defining a distribution on $\mathbb{R}^k\setminus 0$. Then it is simply a question of showing that there is a uniquely defined extension to all of $\mathbb{R}^k$.

Exercise 3.3.15.

(1) Show that if $a(\xi)$ is asymptotically homogeneous of degree $m<0$ then the associated distribution $u(z)$ satisfies

$$\lim_{\epsilon\to 0}<u,\psi(\epsilon^{-1}z)\phi(z)>=0,$$

for any function $\psi\in\mathcal{C}_c^\infty(\mathbb{R}^k)$. hint: use the fact that

$$(1+|\xi-\eta|)(1+|\xi|)>(1+\frac{1}{2}|\eta|).$$

(2) Using part 1 show that Lemma (3.3.13) extends to this case with no modification.

(3) Can you figure out what to do in the general case?

More generally we can consider the distributions defined by symbols $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^k)$ which are homogeneous in ξ for $|\xi| > 1$. The symbol depends smoothly upon the x–variables and differentiating in x does affect the homogeneity properties. Thus the foregoing calculations apply without essential modification to show that the distribution defined by the oscillatory integral

$$u(x,y) = \int e^{i\xi\cdot y} a(x,\xi) d\xi$$

has an asymptotic expansion. For simplicity we again assume that $m < -k$ so the integral converges absolutely. Setting $y = r\omega$, $|\omega| = 1$ we have

$$u(x,r\omega) = r^{-(m+k)}(f(x,\omega) + \log r g(x,r\omega)) + h(x,r\omega), \tag{3.3.16}$$

where $f(x,\omega)$ is a smooth function on $\mathbb{R}^n \times \mathbb{S}^{k-1}$, $g(x,y), h(x,y)$ are smooth in $\mathbb{R}^n \times \mathbb{R}^k$ with $g(x,y) = O(|y|^{-(m+k)})$.

The expansion in (3.3.16) is in many ways the main point of this section. What it shows is that if we introduce polar coordinates about the submanifold $y = 0$, then a conormal distribution with an asymptotically homogeneous symbol has a very simple sort of a singularity. Essentially we get functions smooth away from $r = 0$ with a simple algebraic behavior as $r \to 0$ and smooth in all other directions. We see that if we apply differential operators which are polynomials in the vector fields

$$\{r\partial_r, X_1, \ldots, X_{k-1}, \partial_{x_1}, \ldots, \partial_{x_n}\},$$

where $X_i, i = 1, \ldots, k-1$ are tangent to the sphere, to $u(x,r,\omega)$ then we get a function with precisely the same regularity as u. We make this more precise in the next section.

If $A \in \Psi^m_{\mathrm{KN}}$ then we can use (3.3.16) to obtain an asymptotic expansion for the Schwarz kernel. As before, we assume for simplicity that $m < -n$ though one can easily generalize this to all orders. The expansion is

$$K_A(x, x-y) \sim \sum_{j=0}^{\infty} k_j(x,\omega)|x-y|^{j-m-n} + |x-y|^{-(m+n)} \log|x-y| k_0'(x, x-y) + l(x, x-y).$$

As before $k_j(x,\omega)$ are smooth on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ and $k_0'(x,z), l(x,z)$ are smooth in $\mathbb{R}^n \times \mathbb{R}^n$ with $k_0'(x,z) = O(|z|^{-(m+k)})$.

We need conormal distributions in one other slightly different situation. In the analysis of the Bergman laplacian we need to consider distributions on manifolds with boundaries or more generally, manifolds with corners. The model space is denoted by $\mathbb{R}^n_k$. It is defined by

$$\mathbb{R}^n_k = \{x \in \mathbb{R}^n; 0 \le x_i, i = 1, \ldots, k\}.$$

This is the model for the intersection of k–boundary components. First we consider the simple case of a manifold with boundary, this is modeled on $\mathbb{R}^n_1$.

On $\mathbb{R}^n_1$ there are two naturally defined spaces of smooth functions, the functions smooth up to the boundary $\mathcal{C}^\infty(\mathbb{R}^n_k)$ and the ideal of functions which vanish to infinite order at the boundary, $\dot{\mathcal{C}}^\infty(\mathbb{R}^n_1)$. We consider distributions in the dual of the latter space. These are called extendible distributions as we can include $\dot{\mathcal{C}}^\infty(\mathbb{R}^n_1)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$ as a closed subspace. Therefore by the Hahn–Banach theorem any continuous linear functional on $\dot{\mathcal{C}}^\infty(\mathbb{R}^n_1)$ has an extension to $\mathcal{C}^\infty(\mathbb{R}^n)$. We denote this space by $\mathcal{C}^{-\infty}(\mathbb{R}^n_1)$.

As usual we are not interested in the general element of $\mathcal{C}^{-\infty}(\mathbb{R}^n_1)$ but only a very special subspace. The polyhomogeneous conormal distributions of 'order' α, where $\alpha \in \mathbb{C}, m \in \mathbb{N}$, $I^{(\alpha,m)}(\mathbb{R}^n_1; \partial\mathbb{R}^n_1)$ are those elements of $\mathcal{C}^{-\infty}(\mathbb{R}^n_1)$ which have an expansion along $x_1 = 0$ of the form

$$u(x_1, x') \sim \sum_{l=0}^{m}\sum_{j=0}^{\infty} u_{jm}(x') x_1^{\alpha+j}(\log x_1)^l. \tag{3.3.17}$$

Here the functions $u_{jm}(x')$ are assumed to belong to $\mathcal{C}^\infty(\partial\mathbb{R}^n_1)$. More generally we assume that u can be expressed as a sum of such expansions with different exponents $\{\alpha_k, m_k\}$ so long as $\operatorname{Re}\alpha_j$ tends to ∞ as $k \to \infty$. The collection of numbers

$$\mathcal{A} = \{\alpha_k + j, m_k, k = 1, \dots, \infty, j \in \mathbb{N}_0\}$$

is called an index set. The conormal distributions with this index set are denoted by $I^{\mathcal{A}}(\mathbb{R}^n_1; \partial\mathbb{R}^n_1)$.

For the case of $\mathbb{R}^n_k$ we define polyhomogeneous conormal distributions relative to $\partial\mathbb{R}^n_k$, in the simplest, case as those with an expansion

$$u(x', x'') \sim \sum_{j \in \mathbb{N}_0^k} r^{\alpha + j} u_j(x''). \tag{3.3.18}$$

Here $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$ and $u_j(x'')$ are smooth functions on $\mathbb{R}^{n-k}$. As with a single boundary component we can augment this expansion with log terms and then combine them with different k–tuples α_m. The asymptotic summation makes sense so long as

$$\min_i \operatorname{Re}\alpha_{im}$$

tends to ∞ as $m \to \infty$.

These distributions are conormal in the previous sense as well. It is clear that the expansion is essentially unchanged if we apply a differential operator in the span of

$$\{x_1\partial_{x_1}, \dots, x_k\partial_{x_k}, \partial_{x_{k+1}}, \dots, \partial_{x_n}\}. \tag{3.3.19}$$

Here we have taken the intersection of all the vector fields tangent to each of the hypersurface boundary faces which meet at $x = 0$.

There are several important things to observe about these distributions, firstly they form a ring. In suggestive notation we have

$$I^{\mathcal{A}}(\mathbb{R}^n_k; \partial\mathbb{R}^n_k) \cdot I^{\mathcal{B}}(\mathbb{R}^n_k; \partial\mathbb{R}^n_k) \subset I^{\mathcal{A}+\mathcal{B}}(\mathbb{R}^n_k; \partial\mathbb{R}^n_k) \tag{3.3.20}$$

and

$$I^{\mathcal{A}}(\mathbb{R}^n_k; \partial\mathbb{R}^n_k) + I^{\mathcal{B}}(\mathbb{R}^n_k; \partial\mathbb{R}^n_k) \subset I^{\mathcal{A}\cup\mathcal{B}}(\mathbb{R}^n_k; \partial\mathbb{R}^n_k). \tag{3.3.21}$$

As we do not have the time we must simply leave these in their suggestive form.

As a final observation we note that such a distribution has a local Mellin transform for $\operatorname{Re}\xi_i$ sufficiently large.

$$\widehat{u}(\xi; x'') = \int\!\!\cdots\!\!\int_{x_i > 0, i = 1, \dots, k} u(x', x'')\phi(x')x^\xi \frac{dx_1 \dots dx_k}{x_1 \dots x_k}. \tag{3.3.22}$$

Here $\phi(x')$ is a smooth function supported in a neighborhood of 0. In fact any extendible distribution has a Mellin transform for sufficiently large $\operatorname{Re}\xi_i$. However if the distribution is polyhomogeneous then the Mellin transform has a meromorphic extension to $\mathbb{C}^k$. The principle parts at the singularities have expansions with coefficients in $\mathcal{C}^\infty(\mathbb{R}^{n-k})$. Though we do not have time to prove it, this property provides a very useful characterization of polyhomogeneous conormal distributions.

3.4 A simple model problem for blow–ups

In this section we introduce a geometric construction which is crucial in the analysis of the singularities occurring in the resolvent of the Bergman Lapalcian. First we consider a very simple model problem: the operator $(x\partial_x)^2$ on the $\mathbb{R}_+$. We would like to construct the Schwarz kernel for the resolvent of the operator that is $((x\partial_x)^2 - s^2)^{-1}$. It is elementary to show that the kernel is given by

$$R(x, y; s) = \begin{cases} \left(\frac{x}{y}\right)^s & \text{if} \quad x < y \\ \left(\frac{y}{x}\right)^s & \text{if} \quad y < x. \end{cases} \tag{3.4.1}$$

The kernel is defined on the manifold with corners $\mathbb{R}^2_2$. As a distribution it is polyhomogeneous conormal along three submanifolds, the diagonal, Δ, the left boundary $x = 0$ which we denote by lb and the right boundary $y = 0$, which we denote by rb. Note however that it fails to be conormal where all three intersect in the corner of the diagonal $\partial\Delta$. It would be very hard to analyze the singularity of this operator $\partial\Delta$ and even harder to try to construct such an operator without having a formula. For example it would be quite difficult to construct a parametrix for the resolvent kernel for a simple perturbation of our operator such as $(x\partial_x)^2 + q(x)$.

Using a simple geometric construction we can make these difficulties vanish. We construct a new manifold with corners by blowing up the boundary of the diagonal. Simply put this amounts to the introduction of polar coordinates about this submanifold. Proceeding a little more formally we define the normal bundle to $\partial\Delta$ by

$$N\partial\Delta = T\mathbb{R}^2_2 \restriction_{\partial\Delta} / T\partial\Delta. \tag{3.4.2}$$

The functions x, y are defining functions for $\partial\Delta$. Note that dx, dy are well defined on $N\partial\Delta$ as they annihilate $T\partial\Delta$. We define the inward pointing normal bundle to $\partial\Delta$, $N_+\partial\Delta$, as the subset of $N\partial\Delta$ where both dx and dy are non-negative. Loosely speaking, the flow for a short time along these 'directions' carries you into the manifold.

On the inward pointing bundle there is a natural action by $\mathbb{R}_+$ induced form the vector space structure of $T\mathbb{R}^2_2$. We define the blowup of $\mathbb{R}^2_2$ along the submanifold $\partial\Delta$ as a topological space by

$$\mathbb{R}^2_{2,0} = \mathbb{R}^2_2 \setminus \partial\Delta \amalg N_+\partial\Delta/\mathbb{R}_+. \tag{3.4.3}$$

There is a naturally defined projection map

$$\beta_0 : \mathbb{R}^2_{2,0} \longrightarrow \mathbb{R}^2_2$$

called the blowdown. It is one to one on $\mathbb{R}^2_2 \setminus \partial\Delta$. The inverse image of 0 is a new boundary component which we call the front face, ff .

We would like to give this the structure of a manifold with corners. We need to define the ringe of smooth functions. There is an identification between a neighborhood of the zero section in $N_+\partial\Delta$ and a neighborhood of $\partial\Delta$ in $\mathbb{R}^2_2$. In this simple case this identification can be done globally but this is not required. Let Ψ denote such an identification. Let M_δ denote the action of $\mathbb{R}_+$ on $N_+\partial\Delta$. Then

$$\widetilde{M_\delta} = \Psi \circ M_\delta \circ \Psi^{-1}$$

defines an action of a neighborhood of 0 in $\mathbb{R}_+$ on a neighborhood of $\partial\Delta$ in $\mathbb{R}^2_2$. We define let CI_0 denote the ring of functions defined on $\mathbb{R}^2_2 \setminus \partial\Delta$ generated by smooth functions on $\mathbb{R}^2_2$ and smooth functions on $\mathbb{R}^2_2 \setminus \partial\Delta$ homogeneous of degrees 0 and 1 with respect to $\widetilde{M_\delta}$. Such functions obviously have well defined lifts to $\mathbb{R}^2_{2,0}$ which extend continuously to ff . We define

$$\mathcal{C}^\infty(\mathbb{R}^2_{2,0}) = \beta_0^*(\mathcal{C}^\infty_0).$$

One can show fairly easily that this is well defined independent of the choice of Ψ. This completes the construction of $\mathbb{R}^2_{2,0}$. We define the lift of the diagonal as

$$\Delta_0 = \overline{\beta_0^{-1}(\Delta \setminus \partial\Delta)}.$$

In this simple case we can express everything in terms of simple concepts. When we use

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \; r = \sqrt{x^2+y^2}$$

the blown–up space is simply

$$\mathbb{R}^2_{2,0} = [0, \frac{1}{2}\pi] \times [0, \infty).$$

A function on $\mathbb{R}^2_2$ lifts to be smooth if it is a smooth function of these variables. For example

$$R_0(s) = \beta_0^*(R(s)) = \begin{cases} \tan\theta^s \text{ if } 0 \le \frac{1}{4}\pi \\ \tan\theta^{-s} \text{ if } \frac{1}{4}\pi \le \frac{1}{2}\pi. \end{cases} \tag{3.4.4}$$

In the blown the analysis of $R_0(s)$ is a lot simpler.

The main reason for this is that the left and right boundaries of $\mathbb{R}^2_{2,0}$ defined by

$$\mathrm{lb} = \overline{\beta_0^{-1}(\mathrm{lb} \setminus \mathrm{lb} \cap \partial\Delta)}, \mathrm{rb} = \overline{\beta_0^{-1}(\mathrm{rb} \setminus \mathrm{rb} \cap \partial\Delta)},$$

no longer intersect the lifted diagonal Δ_0. Note that R_0 has a smooth extension to the interior of the ff away from $\Delta_0 \cap \mathrm{ff}$. In the blown–up space the lifted resolvent kernel is a polyhomogeneous conormal distribution with respect to the submanifolds $\mathrm{lb}, \mathrm{rb}, \Delta_0$.

Before turning to a more general context we consider the front face in greater detail. This boundary component is a fibration over $\partial\Delta$. This is essentially immediate from the construction as the front face is defined as

$$\mathrm{ff} = N_+\partial\Delta/\mathbb{R}_+.$$

Of course in this trivial case there is only one fiber but we won't take further notice or advantage of this fact. Note that the lifted diagonal intersects each fiber in exactly one point. The vector field $x\partial_x$ lifts to $\mathbb{R}^2_2$ in an obvious way. What is a little less obvious is that it also lifts to $\mathbb{R}^2_{2,0}$. This is intuitively reasonable as the new 'directions' in the blown–up space are directions in which $x\partial_x$ vanished in $\mathbb{R}^2_2$. To see this clearly we use projective coordinates for $\mathbb{R}^2_{2,0}$ defined by

$$t = \frac{x}{y} \text{ and } y.$$

These define a coordinate system away from rb, the projection is given by

$$\beta_0(t, y) = (ty, y).$$

A simple calculation shows that

$$\beta_{0*} t\partial_t = x\partial_x. \tag{3.4.5}$$

In other words $x\partial_x$ lifts to a smooth vector field on $\mathbb{R}^2_{2,0}$ which is tangent to the fibers of the front face and is transverse to the lifted diagonal, $t = 1$.

Of course the lifted resolvent kernel also has a restriction to the front face. It is as a distribution with polyhomogeneous conormal singularities along the intersection of the front face with Δ_0 and the front face with the left and right boundaries. We call this restriction the 'normal operator' of the lifted resolvent. It is a new type of symbol. We denote it by $N(R)$. A moments thought shows that

$$N((x\partial_x)^2 R) = (t\partial_t)^2 N(R). \tag{3.4.6}$$

From (3.4.6) we deduce that the resolvent equation

$$((x\partial_x)^2 - s^2)R(s) = \delta_\Delta$$

lifts and restricts to the front face to give

$$((t\partial_t)^2 - s^2)N(R_0(s)) = \delta_{\Delta_0 \cap \mathrm{ff}}. \tag{3.4.7}$$

In other words the normal operator of $R_0(s)$ is itself defined by an equation along the fibers of the front face. It is difficult to explain the significance of this fact in so simple an example.

In the analysis of the Bergman Laplacian something a little more general than a linear blowup, as described above is needed. More background is required before we can really explain the why this is the case. For now we consider the problem of trying to resolve the singularity of a function defined on $\mathbb{R}^2_2$. Let

$$f(x,y) = \frac{x^2}{y},$$

and observe that introducing polar coordinates does not resolve the singularity of this function at $x = y = 0$:

$$\beta^* f(r,\theta) = \frac{r\cos\theta}{\sin^2\theta}. \tag{3.4.8}$$

From (3.4.8) it is apparent that

$$\lim_{r\to 0,\theta\to 0} \beta^* f(r,\theta)$$

depends upon the direction of approach. This is precisely what is meant by the statement that the blowup did not resolve the singularity.

A slightly different way to describe the blowup process is to say that we have an action by $\mathbb{R}_+$ and we remove the fixed points of this action replacing them with one fixed point for each trajectory of the $\mathbb{R}_+$ action. Clearly the distributions whose singularities are resolved by the blow–up are those with homogeneity properties relative to the $\mathbb{R}_+$–action. This view explains why the singularity of f is not resolved by a blowup whose underlying $\mathbb{R}_+$–action is

$$M_\delta(x,y) = (\delta x, \delta y).$$

It also suggests that we replace this action with a different action, better adapted to the behavior of f near to $(0,0)$.

Define a new $\mathbb{R}_+$–action on $\mathbb{R}^2_2$ by

$$N_\delta(x,y) = (\delta x, \delta^2 y).$$

Observe that

$$N_\delta^* f = f;$$

in other words f is homogeneous of degree 0 relative to this new action. Using this new action we define a parabolic blowup of $\mathbb{R}^2_2$. This space is defined by

$$\mathbb{R}^2_{2,(0,dy)} = \mathbb{R}^2_2 \setminus \partial\Delta \amalg N_+\partial\Delta/\mathbb{R}_+, \tag{3.4.9}$$

where the $\mathbb{R}_+$ is defined by N_δ instead of M_δ. Topologically the two blown–up spaces are identical. They differ in their $\mathcal{C}^\infty$–structure. As before we have a blow–down map

$$\beta_{0,dy} : \mathbb{R}^2_{2,(0,dy)} \longrightarrow \mathbb{R}^2_2.$$

We define the $\mathcal{C}^\infty$–structure on $\mathbb{R}^2_{2,(0,dy)}$ as the ring of functions generated by $\beta^*_{0,dy}\mathcal{C}^\infty(\mathbb{R}^2_2)$ and the lifts of smooth functions on $\mathbb{R}^2_2 \setminus (0,0)$ which are homogeneous of degrees 0 and 1 with respect to the $\mathbb{R}_+$–action defined by N_δ.

To define local coordinates we set

$$r^4 = x^4 + y^2;\ \alpha = \frac{x}{r}, \beta = \frac{y}{r^2}.$$

These functions are clearly smooth on the blown–up space being homogeneous of degrees 1 and 0 respectively. Note that

$$\alpha^4 + \beta^2 = 1.$$

In terms of these coordinates we can lift f:

$$\beta^*_{0,dy} f = \frac{\alpha^2}{\beta} = \frac{\sqrt{1-\beta^2}}{\beta}.$$

This function is of course infinite where $\beta = 0$ however it is 'single valued' and in fact a polyhomogeneous conormal distribution on the blown–up space.

3.5 Parabolic blow–ups for the model problem

In the previous section we discussed how blowups could be used to resolve certain kinds of singularities that arise in the study of extendible distributions. In some ways we were putting the cart before the horse as the question which we would like to answer is what sort of singularities arise in the construction of the resolvent kernel for the Bergman Laplacian. The data we are given is the operator not the kernel of its inverse. As is customary in the analysis of operators it is better to work with an algebra of operators rather than a single operator. What we shall see is that the algebra of operators dictates how to define the blowup and then the relevant kernels will be desingularized, as if by magic.

On a compact manifold the usual algebra is the algebra generated over the smooth functions by the vector fields. What we need is a reasonable substitute for vector fields, which are of course, the sections of the tangent bundle. Since we want the Laplace operator to belong to this algebra and to be an elliptic element, a good possibility would be the algebra generated by sections of $T\Omega$ which are of uniformly bounded length relative to the Bergman metric and smooth up to the boundary. To examine these vector fields it is easier to replace the unit ball by a biholomorphically equivalent model, the upper half space defined by the hyperquadric.

The hyperquadric is defined by

$$\mathcal{Q} = \{(w, z) \in \mathbb{C}^n; \operatorname{Im} w = \frac{1}{2}|z|^2\}. \tag{3.5.1}$$

The upper half space is given by

$$\mathcal{Q}^+ = \{(w, z) \in \mathbb{C}^n; \operatorname{Im} w > \frac{1}{2}|z|^2\}.$$

I leave it as an exercise to find the bi holomorphic mapping from $\mathbb{CB}^n$ onto $\mathcal{Q}^+$. If we use this mapping to pullback the defining function we see that we get

$$|h(w, z)|^2(\operatorname{Im} w - \frac{1}{2}|z|^2), \tag{3.5.2}$$

here $h(w, z)$ is a holomorphic function. So pulling back the Bergman metric we get a metric defined on $\mathcal{Q}^+$ by

$$g_{\mathrm{B}} = -\partial\overline{\partial} \log(\operatorname{Im} w - \frac{1}{2}|z|^2).$$

If we let $\rho = \operatorname{Im} w - \frac{1}{2}|z|^2$ then the metric has the simple form in these coordinates

$$g_{\mathrm{B}} = \frac{1}{2}\frac{dz_i d\bar{z}_i}{2\rho} + \frac{\theta}{\rho} \wedge \frac{\bar{\theta}}{\rho}. \tag{3.5.3}$$

where

$$\theta = \partial\rho = idw + \bar{z}_i dz_i. \tag{3.5.4}$$

A unit basis of vector fields in this coordinate system is given by

$$W = 2\rho\partial_w, Z_i = \sqrt{2\rho}(\partial_{z_i} + i\bar{z}_i\partial_w), i = 1, \ldots, n-1. \tag{3.5.5}$$

These display a slightly alarming property, the last $n-1$ are not actually smooth up to the boundary but have a square root singularity. This leaves us little choice but to change the $\mathcal{C}^\infty$–structure of the domain itself at the boundary. We extend the ring of $\mathcal{C}^\infty$–functions by adjoining the square root of a smooth defining functions. We denote by $\mho$ the domain Ω with the $\mathcal{C}^\infty$–structure defined as $\mathcal{C}^\infty(\Omega)[\sqrt{\rho}]$. This family of functions is of course defined independently of the choice of defining function.

We introduce new coordinates $r = \sqrt{\rho}, u = \operatorname{Re} w$, and the old $z_i, i = 1, \ldots, n-1$. A function is smooth at $\partial\mho$ if it has a Taylor expansion

$$\begin{aligned} f &\sim \sum_{j=0}^{\infty} f_j(u, z, \bar{z}) r^j \\ &\sim \sum_{j=0}^{\infty} f_j(u, z, \bar{z}) \rho^{\frac{1}{2}j} \end{aligned} \tag{3.5.6}$$

where $f_j(u, z, \bar{z}), j = 0, \ldots$ are smooth functions of their arguments. Notice that $\partial\Omega$ and $\partial\mho$ are canonically isomorphic as smooth manifolds it is only the way in which they are attached to the interior which has changed.

We compute the real and imaginary parts of the basis in (3.5.5) relative to the $r, u, z_i, \bar{z}_i$–coordinates:

$$\begin{aligned} T = \operatorname{Re} W = r^2\partial_u, -N = \operatorname{Im} W = -\frac{1}{2} r\partial_r, \\ X_i = \operatorname{Re} Z_i = \frac{r}{\sqrt{2}}(\partial_{x_i} + y_i\partial_u), -Y_i = \operatorname{Im} Z_i = -\frac{r}{\sqrt{2}}(\partial_{y_i} - x_i\partial_u). \end{aligned} \tag{3.5.7}$$

Notice that the vector fields are smooth in the square root differential structure. These vector fields define a finite dimensional Lie algebra, the non–zero brackets are:

$$\begin{aligned} [N, T] = 2T, [N, X_i] = X_i, [N, Y_i] = Y_i; \\ [X_i, Y_i] = \delta_{ij} T. \end{aligned} \tag{3.5.8}$$

The vector fields T, X_i, Y_i define the Heisenberg algebra, and N defines a homogeneous extension of this algebra. Let $N_\delta(p)$ denote the flow on $\mathcal{Q}^+$ defined by

$$N_\delta(r, u, x, y) = (\delta r, \delta^2 u, \delta x, \delta y). \tag{3.5.9}$$

The basis of vector fields defined in (3.5.7) is invariant under this flow. In other words, the vector fields are homogeneous of degree zero with respect to the this action.

Exercise 3.5.10**.** Find the formula for the Laplace operator of the Bergman metric relative to the basis in (3.5.7).

We can use this homogeneity structure to blow–up the point 0 on the boundary of $\mathcal{Q}^+$. Observe that if we set

$$\Theta = du + \frac{1}{2}\sum x_i dy_i - y_i dx_i, \tag{3.5.11}$$

Then the directions in which we scale linearly are precisely the kernel of Θ. The dilation structure is obtained by integrating the vector field

$$R = r\partial_r + 2u\partial_u + x \cdot \partial_x + y \cdot \partial_y.$$

Denote the blowdown by $\beta_{0,\Theta}$. Coordinates are given by

$$R^4 = u^2 + (r^2 + |z|^2)^2, \alpha = \frac{u}{r^2}, \beta = \frac{r}{R}, \zeta_i = \frac{z_i}{R}. \tag{3.5.12}$$

A simple computation shows that the Bergman kernel pulled back to the $\mathcal{Q}^+$ is given by

$$B(w, z; \tau, \xi) = \frac{2}{(i(w - \overline{\tau}) - z \cdot \overline{\xi})^{n+1}}. \tag{3.5.13}$$

A moments thought shows that if we fix $\tau, \xi \in \partial\mathcal{Q}^+$ then the kernel $B(w, z; \tau, \xi)$ has a very complicated singularity as we approach τ, ξ. If we pull back $B(w, z; 0, 0)$ via the blowdown map the picture is greatly simplified:

$$\beta_{0,\Theta}^*(B(\cdot; 0, 0)) = \frac{\alpha - i(\beta^2 + |\zeta|^2)}{R^{2(n+1)}}. \tag{3.5.14}$$

The pulled back kernel is a smooth function in the blown–up coordinates times a power of the defining function of the front face. Evidently the blow–up resolved the singularity of the kernel at the point $0, 0$. The choice of the point is arbitrary as there is a transitive group of isometries, so we could do an analogous construction at any point of $\partial\mathbb{CB}^n$. In general we can decide which direction scales linearly by consideration of the one form Θ defined in (3.5.11). At each point $(0, u, x, y) \in \partial\mathcal{Q}^+$ the kernel of Θ consists of the complex tangent directions to $\partial\mathcal{Q}^+$ along with the vector field N. The complementary direction is given by T which is simply the almost complex structure applied to N. Recall that before we introduced the square root differential structure the vector field $W = T + iN$ behaved like $O(\rho)$ at the boundary whereas the other directions behaved like $O(\sqrt{\rho})$. Thus we see that the parabolic homogeneity structure, defined at each point of $\partial\mathcal{Q}^+$ by scaling linearly in direction belonging to $\ker\Theta$ and parabolically in the complementary direction, is the same structure as that defined by the Bergman kernel at the boundary. In fact we shall later see that it is the structure defined by the one form which is primary and the metric is really secondary.

Of course we really want to do this at all points of the $\partial\mathbb{CB}^n$ simultaneously. This means we should work in $\mathbb{CB}^n \times \mathbb{CB}^n$ instead. We want to blow up the locus where the Bergman kernel is singular and apparently we need to blow it up parabolically. The locus we need to blow up is the $\partial\Delta$. The question is how should we decide which directions to scale linearly and which directions to scale quadratically. In the examples we've already considered, we had a one form which helped to make this determination. In this case we also have a one form.

It arises as follows, the space $\mathbb{CB}^n \times \mathbb{CB}^n$ has two projections down to $\mathbb{CB}^n$ we denote them by π_l and π_r. In the models considered in the previous section we needed to consider a homogeneity defined on the the inward pointing normal bundle to the submanifold we wanted to blow–up. In the case at hand this is $N_+\partial\Delta$. We need to define a one form on this bundle. Let

$$\widetilde{\Theta} = \pi_l^*(\Theta) - \pi_r^*(\Theta).$$

This one form vanishes identically when restricted to the Δ and therefore it defines a one form on $N_+\partial\Delta$. We use it to define the parabolic blowup.

In the local coordinates introduced above the one form is given by

$$\widetilde{\Theta} = d(u - u') + \frac{1}{2}\sum_i (x_i - x_i')dy_i + x_i' d(y_i - y_i') - (y_i - y_i')dx_i - y_i' d(x_i - x_i'). \tag{3.5.15}$$

Here and in the sequel we use $'$ to indicate the coordinates on the 'other' copy of $\mathbb{CB}^n$. In order to have a coordinate to scale parabolically we need to write $\widetilde{\Theta} = dt$ for some function t. This only needs to hold along $\partial\Delta$, set

$$t = u - u' + \frac{1}{2}\sum_{j=1}^{n-1} x_j(y_j - y_j') - y_j(x_j - x_j'). \tag{3.5.16}$$

We define the parabolic blowup by introducing the "polar" coordinates:

$$\begin{aligned} R^4 =&(r^2 + r'^2 + \frac{1}{2}\sum_{j=1}^{n-1}|z_i - z_i'|^2)^2 + t^2 \\ \rho_{\mathrm{lb}} =&\frac{r}{R}, \rho_{rb} = \frac{r'}{R}, T = \frac{t}{R^2}, Z_i = \frac{z_i - z_i'}{\sqrt{2}R}. \end{aligned} \tag{3.5.17}$$

If one expresses the Bergman kernel in these coordinates one obtains

$$\beta^*_{\partial\Delta,\Theta}(B) = \left[\frac{\rho_{\mathrm{lb}}^2 + \rho_{\mathrm{rb}}^2 + |Z|^2 - iT}{\rho_{\mathrm{lb}}\rho_{\mathrm{rb}}}\right]^{2(n+1)}$$

Let us examine more closely the space obtained by blowing up $\partial\Delta$. We denote the blown–up space by $[\mathbb{CB}^2_{\frac{1}{2}}]_{\partial\Delta,\Theta}$. The $\frac{1}{2}$ indicates that we use the square root differential structure. As in the one dimensional case $[\mathbb{CB}^2_{\frac{1}{2}}]_{\partial\Delta,\Theta}$ has three boundary components, a left and right boundary, coming from the unblownup space and a front face introduced by the blowup. The front face is abstractly defined as

$$N_+(\partial\Delta)/\mathbb{R}_+,$$

where $\mathbb{R}_+$ acts 'parabolically'. Since $N_+(\partial\Delta)$ is a bundle over $\partial\Delta$ with fibers isomorphic to $\mathbb{R}^{n+1}_2$ and the $\mathbb{R}_+$–action preserves the fibers we conclude that the front face is canonically a bundle over $\partial\Delta$ with fibers isomorphic to $\mathbb{S}^n \cap \mathbb{R}^{n+1}_2$. Such a bundle is called a quarter sphere bundle.

The inward pointing tangent bundle to the diagonal is a subbundle of the inward pointing tangent bundle to the whole space. The image of this subbundle in $N_+(\partial\Delta)$ is a half line bundle over $\partial\Delta$. The half line is located in the interior of the fiber. In the front face the quotient of this subbundle defines a distinguished point in each fiber. This is the locus where the lifted diagonal intersects the front face. Note that this locus is in the interior of the fiber of the front face and therefore, in the blown–up space, the lifted diagonal does not meet the left and right boundaries. This was one of the primary goals of introducing a blown–up space.

A very important feature of the parabolic blowup is the way in which the unit basis lifts to the blown–up space. We compute the lift of the basis given in (3.5.7) from the left, in term of projective coordinates. A good coordinate system is given by

$$\begin{gathered} r', u', z_j'; \\ \rho = \frac{r}{r'}, \tau = \frac{t}{r'^2}, \xi_j + i\eta_j = \frac{z_j - z_j'}{r'}. \end{gathered} \tag{3.5.18}$$

In terms of this coordinates we have

$$N \rightsquigarrow \rho\partial_\rho,\ T \rightsquigarrow \rho^2\partial_\tau,\ X_j \rightsquigarrow \rho(\partial_{\xi_j} + \eta_j\partial_\tau),\ Y_j \rightsquigarrow \rho(\partial_{\eta_j} - \xi_j\partial_\tau). \tag{3.5.19}$$

Denote the lifted vector fields with a tilde, e.g. $\widetilde{T}$. The lifted vector fields are clearly tangent to the fibers of the front face. Moreover they span the tangent space to the fiber and are transverse to the diagonal.

What is immediately apparent from (3.5.19) is that the lifted vector fields have exactly the same structure as the unlifted vector fields. In the case at hand this should not be a surprise as the basis of vector fields is invariant under a transitive group action. One interprets (3.5.19) as saying that the lie algebra is actually homogeneous in an infinitesimal sense at the boundary. In fact we shall see that this structure would be present at the front face even if we started with a basis which was given by $\mathcal{C}^\infty$–combinations of the vector fields in (3.5.7). This is also easy to explain, the lift of a vector field to the fiber over a point $p \in \partial\Delta$ depends only on the germ of that vector field at the point p itself. Thus if

$$V = a(q)T + b(q)N + c_j(q)X_j + d_j(q)Y_j,$$

then the lifted vector field in the fiber over p is simply

$$\widetilde{V} = a(p)\widetilde{T} + b(p)\widetilde{N} + c_j(p)\widetilde{X}_j + d_j(p)\widetilde{Y}_j.$$

The coefficients are constant along the fiber.

This indicates that the fibers of the front face have the structure of a lie group defined by lifting a basis of smooth vector fields. The lifted vector fields, restricted to a fiber, define the left invariant vector fields and the intersection of the diagonal with the fiber defines the identity element in the group. So far all our lifts have been from the left, an analogous construction can be done lifting form the right. One simply gets the group acting on the right instead of the left. The transposition

$$\mathcal{T} : [\mathbb{CB}^n]^2 \longrightarrow [\mathbb{CB}^n]^2$$

defined by $\mathcal{T}(p,q) = (q,p)$ lifts to a smooth map of the blownup space to itself. This maps conjugates the left action to the right.

The left lift of vector fields can be extended in a unique way to differential operators which can be expressed as polynomials in this basis with $\mathcal{C}^\infty$–coefficients. The remarks above indicate that if $P(q,T,N,X,Y)$ is such a differential operator, then its lift to the fiber of the front face over p is of the form $P(p,\widetilde{T},\widetilde{N},\widetilde{X},\widetilde{Y})$. In other words, the restriction to a fiber of the front face gives a left invariant differential operator relative to the natural lie group structure.

In addition to lifting differential operators we can also lift a unit frame and thereby define a Riemannian metric on each fiber, which varies smoothly from fiber to fiber. The lift of a basis of $T^{1,0}$–vector fields defines an integrable almost complex structure on the fibers which also varies smoothly from point to point.

The fiber is a quarter sphere; it is important to remember that the fibers are compact manifolds with corners. The two boundary components of the fiber arise from the intersections of the fiber with the left and right boundaries of $[\mathbb{CB}^2_{\frac{1}{2}}]_{\partial\Delta,\Theta}$. The fiber over the point $p \in \partial\mathbb{CB}^n$ can be identified with the the unit ball blown up at the point p precisely as described at the beginning of this section. There is a small degree of arbitrariness in the way this is done because there is no distinguished point in the unit ball whereas each fiber has a distinguished point. Once one chooses a point in the blown–up ball then you need to choose an isomorphism of the lie algebra of left invariant vector fields at that point with those at the identity element in the fiber of the front face. This can be done smoothly in the base point and any two choices are canonically isomorphic. One extends the map by integrating the vector fields. An interesting feature of the ball blown–up at a boundary point, is that the group action defined by the left invariant vector fields extends continuously to the compactification defined by the blow–up.

3.6 The Θ–tangent bundle and parabolic blow–ups.

At this point we are ready to generalize the construction carried out on the unit ball to a smooth strictly pseudoconvex domain, Ω. When analyzing the Laplace operator defined by a metric on a compact manifold we really begin with the lie algebra of smooth vector fields and then construct the universal enveloping algebra. The Laplace operator of any metric belongs to this algebra. The algebra of pseudodifferential operators is a 'quantization' of the lie algebra of vector fields. The main point is that we do not consider a single operator defined by a single metric but rather a $\mathcal{C}^\infty$–lie algebra whose universal enveloping algebra contains the operators we are interested. The first issue we need to settle is how to define a $\mathcal{C}^\infty$–lie algebra which captures the important behavior at the boundary.

This information is entirely encoded by fixing a one form θ at the boundary. Assume that $\theta \restriction_{\partial D} \neq 0$ at any point. We define a vector space of smooth vector fields

$$\mathcal{V}_\theta(D = \{X; \quad X \restriction_{\partial D} = 0, \theta(X) = O(r^2)\},$$

here r is a defining function for the boundary of D. Note that θ must be defined on all of $TD \restriction_{\partial D}$.

Proposition 3.6.1. *The vector space, $\mathcal{V}_\theta(D)$, with the usual lie bracket of vector fields is a $\mathcal{C}^\infty$–lie algebra.*

Proof. If X, Y vanish along ∂D then so does $[X,Y]$. Since the elements of $\mathcal{V}_\theta$ are tangent to ∂D it is easily shown that for $k \geq 0$,

$$Xr^k = O(r^k) \text{ for all } X \in \mathcal{V}_\theta. \tag{3.6.2}$$

The formula of Cartan states that

$$\theta([X,Y]) = X\theta(Y) - Y\theta(X) - d\theta(X,Y). \tag{3.6.3}$$

As $d\theta$ is smooth the last term in (3.6.3) is clearly $O(r^2)$ and $\theta(X), \theta(Y)$ are both $O(r^2)$. The assertion of the proposition therefore follows from (3.6.2).

We would like to use $\mathcal{V}_\theta(D)$ as the replacement for the algebra of vector fields on a compact manifolds. This latter algebra has a crucial property: it is the collection of smooth sections of a vector bundle. In other words it is locally free. The same turns out to be true of $\mathcal{V}_\theta(D)$. To establish this, the easiest thing to do is to write down smooth local trivializations. Since θ restricted to ∂D is non–vanishing we can introduce local coordinates $r, y_1, \dots, y_{2n-1}$ so that

$$\theta = \sum_{j=1}^{2n-1} a_j dy_j. \tag{3.6.4}$$

We can choose a vector field T, tangent to the hypersurfaces r =constant such that $\theta(T) = 1$ and vector fields $X_1, \dots, X_{2(n-1)}$ are also tangent to r =constant with $\theta(X_i) = 0, i = 1, \dots, 2(n-1)$. Evidently the vectors

$$r\partial_r;\ r^2T;\ rX_i, i = 1, \dots, 2(n-1), \tag{3.6.5}$$

belong to $\mathcal{V}_\theta(D)$. If V is any smooth vector field which vanishes on ∂D then V can be expressed as

$$V = a(r)r\partial_r + b(r)rT + \sum_{j=1}^{2(n-1)} c_j(r)rX_j,$$

where $a, b, c_1, \dots, c_{2(n-1)}$ are smooth functions on D. If $V \in \mathcal{V}_\theta$ then evidently $b(r) = O(r)$, thus V is expressible in terms of the basis (3.6.5) with $\mathcal{C}^\infty$–coefficients. This establishes that $\mathcal{V}_\theta(D)$ consists of sections of a smooth vector bundle, ${}^\Theta TD$.

There is a more 'sheaf theoretic' description of this vector bundle. Let $\mathcal{J}_p$ denote the ideal of functions vanishing at $p \in D$. Mimicing the usual construction we define the fiber of our vector bundle by

$${}^\Theta T_pD = \mathcal{V}_\theta / \mathcal{J}_p\mathcal{V}_\theta. \tag{3.6.6}$$

Clearly ${}^\Theta T_pD$ is canonically isomorphic to T_pD if $p \in \overset{\circ}{D}$. However for $p \in \partial D$ things are quite different.

Theorem 3.6.7. *For $p \in \partial D$ the fiber ${}^\Theta T_pD$ of the Θ–tangent bundle is canonically a nilpotent lie algebra. Its structure is determined by the rank of $d\theta \restriction_{T_p\partial D}$.*

Proof. We compute in terms of the local basis given in (3.6.5):

$$\begin{aligned} &[r\partial_r, rX_i] = rX_i + \alpha_i r^T + O(\mathcal{J}_p\mathcal{V}_\theta), [r\partial_r, r^2T] = 2r^2T + O(\mathcal{J}_p\mathcal{V}_\theta); \\ &[rX_i, r^2T] = O(\mathcal{J}_p\mathcal{V}_\theta), [rX_i, rX_j] = \beta_{ij}r^2T + O(\mathcal{J}_p\mathcal{V}_\theta). \end{aligned} \tag{3.6.8}$$

The constants α_i can be removed by replacing rX_i by $rX_i - \frac{1}{2}\alpha_i r^2T$. Thus the structure of the lie algebra is determined by the coefficients β_{ij}. An application of Cartan's lemma shows that

$$\beta_{ij} = -d\theta_p(X_i, X_j). \tag{3.6.9}$$

Since the only invariant of a skew symmetric form on an even dimensional space is its rank, the conclusion of the theorem follows.

Let X_p denote the inward pointing part of the tangent space at $p \in \partial D$. This half space should be thought of as giving a local model for D near to p. The one form θ defines a hyperplane in X_p:

$$H_p = \ker(\theta_p).$$

Let S_p denote a complementary subspace, as noted above this can be taken tangent to the boundary. We use the splitting $X_p = H_p \oplus S_p$ to define a homogeneity on X_p, if $v = h + s, h \in H_p, s \in S_p$ then

$$M_\delta(v) = \delta^1 h + \delta^2 s. \tag{3.6.10}$$

We can choose coordinates at p, $r, y_1, \dots, y_{2(n-1)}, u$ so that $p = (0,0,0)$ and

$$H_p = \operatorname{Span}\{\partial_r, \partial_{y_i} i = 1, \dots, 2(n-1)\}$$

and S_p is spanned by ∂_u. Then the dilation structure in (3.6.10) is the infinitesimal version of

$$M_\delta(r, y, u) = (\delta r, \delta y, \delta^2 u). \tag{3.6.11}$$

We can, without too much confusion, think of r, y, u as giving coordinates in a neighborhood of $p \in \partial D$ and also coordinates for X_p. With this dual interpretation in mind, it follows that for $V \in \mathcal{V}_\theta$

$$V_\delta = M_{\delta^{-1}*} V$$

is a vector field defined in a larger and larger neighborhood of $0 \in X_p$ as δ tends to zero. In fact by expressing V_δ in local coordinates, it is easy to see that

$$\mathfrak{h}_p(V') = \lim_{\delta \downarrow 0} V_\delta$$

is well defined as a smooth vector field on all of X_p. If $a(x)$ is a smooth function in a neighborhood of p and $V \in \mathcal{V}_\theta$, then

$$\mathfrak{h}_p(aV) = a(p)V'. \tag{3.6.12}$$

From this it is apparent that the image of $\mathcal{J}_p \mathcal{V}_\theta$ under $\mathfrak{h}_p$ is zero. Thus we have shown

Proposition 3.6.13. *The map $\mathfrak{h}_p$ defines a lie algebra isomorphism from ${}^\Theta T_p D$ onto a lie algebra of smooth vector fields defined on X_p.*

Proof. It follows immediately from (3.6.12) and the definition that it defines a lie algebra homomorphism. A simple calculation in local coordinates establishes that the map is an isomorphism

Before proceeding we should relate these constructions to the case of a strictly pseudoconvex domain, Ω. In that case we have a naturally defined one form on the boundary ∂r, where r is a defining function. We can extend this one form to all of $T\Omega \restriction_{\partial\Omega}$ by observing that the restriction of

$$\theta = \frac{1}{2i}(\partial r - \overline{\partial} r)$$

to $\partial\Omega$ is agrees with ∂r. However recall from the discussion in the previous section that we needed to introduce the square root differential structure in order for the unit length sections of the tangent bundle to be smooth at the boundary. This is also necessary in the general case. Let $\beta_{\frac{1}{2}}$ denote the blow–down map from the square root differential structure to the standard structure. Locally it is given by

$$\beta_{\frac{1}{2}}(\rho, y) = (\rho^2, y).$$

Lemma 3.6.14. *If θ_1 and θ_2 denote two smooth extension of ∂r to $T\Omega \restriction_{\partial\Omega}$ then*

$$\beta^*_{\frac{1}{2}}\theta_1 \restriction_{\partial\Omega_{/ha}} = \beta^*_{\frac{1}{2}}\theta_2 \restriction_{\partial\Omega_{\frac{1}{2}}} .$$

Proof. Since the difference $\theta_1 - \theta_2$ vanishes on the tangent space to the boundary, we conclude that

$$\theta_1 - \theta_2 = adr + r\alpha,$$

for a smooth function a and one form α. Under the pullback

$$\beta^*_{\frac{1}{2}}(dr) = 2\rho d\rho,$$

from which the conclusion is immediate.

In the sequel we denote Ω with the square root differential structure by $\mho$ and $\Theta = \beta^*_{\frac{1}{2}}(\theta)$. The preceding lemma shows that Θ is determined on all of $T\mho \restriction_{\partial\mho}$ by θ on $\partial\Omega$. On a strictly pseudoconvex domain we define the algebra $\mathcal{V}_\Theta$ relative to this choice of one form. It defines a contact structure on $\partial\mho$, that is

$$\Theta \wedge (d\Theta)^{n-1}$$

does not vanish at any point of $\partial\mho$. This means that the lie algebras ${}^{\Theta}T_p\mho, p \in \partial\mho$, are all isomorphic.

From now on we will consider a defining function r for Ω to be fixed. This in turn defines a metric

$$g = -\partial\overline{\partial}\log r.$$

We say that such a metric is of Bergman type. We also define ρ so that $\rho^2 = r$. We can express the metric on Ω as follows

$$g = \frac{1}{r}\frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} + \frac{\partial r}{r}\frac{\overline{\partial} r}{r}. \tag{3.6.15}$$

From this it follows immediately that a unit basis for this metric which is smooth in the square root differential structure must satisfy

$$\theta(Z_i) = O(r),$$

and therefore, when lifted to $\mho$, these vector fields satisfy

$$\Theta(\widetilde{Z}_i) = O(\rho^2).$$

Thus we have established that

Lemma 3.6.16. *A smooth unit basis for $T^{1,0}\mho$ relative to the metric,* (3.6.15) *consists of local sections of ${}^{\Theta}T\mho$.*

In the proof of the Levi extension theorem we showed that if $p \in \partial\Omega$ then there is a local holomorphic coordinate system, $w, z_1, \ldots, z_{n-1}$ such that

$$\rho(w, z) = \operatorname{Im} w - \frac{1}{2}|z|^2 + O((|z| + |w|)^3), \tag{3.6.16}$$

see (2.9.23). Computing in this coordinate system we see that the metric defined by ρ agrees with the Bergman metric on the unit ball to first order at $(w, z) = (0, 0)$. More precisely, a unit frame in a neighborhood of this point is given by

$$W' = W + O(\mathcal{J}_p\mathcal{V}_\Theta), Z'_i = Z_i + O(\mathcal{J}_p\mathcal{V}_\Theta), \tag{3.6.17}$$

Here W, Z_i is the frame given in (3.5.5). From (3.6.17) it is apparent that if we use Θ to define the local homogeneity structure at $p \in \partial\mho$ then the images $\mathfrak{h}_p(W'), \mathfrak{h}_p(Z_i')$ agree with those obtained by blowing up a point on the boundary of the hyperquadric. We can use the lifted vector fields to define a metric and a complex structure on X_p. It is apparent that with these induced structures, X_p is biholmorphically isometric to the hyperquadric.

We denote by Diff^m_Θ the polynomials in $\mathcal{V}_\Theta$ of degree m with $\mathcal{C}^\infty$–coefficients. Since Diff_Θ is simply the enveloping algebra of the $\mathcal{C}^\infty$–lie algebra, $\mathcal{V}_\Theta$, the homomorphism

$$\mathcal{V}_\Theta \longrightarrow {}^\Theta T_p D$$

extends as a homomorphism of enveloping algebras

$$N_p : \mathrm{Diff}^m_\Theta \longrightarrow \mathcal{D}^m({}^\Theta T_p D).$$

This homomorphism is called the normal operator. Since it is a homomorphism of enveloping algebras it follows that

$$N_p(P \circ Q) = N_p(P) \circ N_p(Q). \tag{3.6.18}$$

We can compose this homomorphism with $\mathfrak{h}$, suitably extended, to obtain a homomorphism from Diff^m_Θ to left invariant operators of order m acting on X_p.

Lemma 3.6.19. *The Laplace operator defined by the metric* (3.6.15) *belongs to* Diff^2_Θ.

Proof. Let $X \in \mathcal{V}_\Theta$ be a smooth vector section and let ϕ_t denote the flow defined by X. Since X is tangent along the boundary the flow is well defined up to $\partial\mho$.

Exercise 3.6.20. Show that if dVol is the volume form of the metric (3.6.15) then

$$\phi_t^*(\mathrm{dVol}) = w_t \,\mathrm{dVol} \tag{3.6.21}$$

where $w_t \in \mathcal{C}^\infty(\mho)$, is differentiable in t.

Using the change of variables theorem we deduce that if $f, g \in \mathcal{C}^\infty_c(\mho)$ then

$$\int\limits_{\mho} fg \,\mathrm{dVol} = \int\limits_{\mho} \phi_t^* f \phi_t^* g w_t \,\mathrm{dVol}\,. \tag{3.6.22}$$

Differentiating (3.6.22) at $t = 0$ and setting $\partial_t w(0) = a$ we obtain that

$$\int\limits_{\mho} Xfg \,\mathrm{dVol} = -\int\limits_{\mho} f(Xg + ag) \,\mathrm{dVol}\,. \tag{3.6.23}$$

The formal symbol of the Laplace operator defined by the metric is determined by the identity:

$$< \Delta f, g >= \int < \overline{\partial} f, \overline{\partial} g > \mathrm{dVol} \;\text{ for } f, g \in \mathcal{C}^\infty_c(\mho). \tag{3.6.24}$$

Since a unit basis for the metric consists of vector fields in $\mathcal{V}_\Theta$ we can express the right hand side of (3.6.24) as

$$\sum_{j=1}^{2n} \int\limits_{\mho} Y_j f \overline{Y_j g} \,\mathrm{dVol},\; Y_j \in \mathcal{V}_\Theta. \tag{3.6.25}$$

The assertion of the lemma follows from (3.6.25) and (3.6.23).

Since $\Delta \in \mathrm{Diff}^2_\Theta$ it has a normal operator.

Proposition 3.6.26. *For a metric of the form* (3.6.15) *the normal operator of the Laplace is the Bergman Laplacian from the unit ball.*

Proof. To prove this statement it is easiest to use the definition of the Laplacian provided by (3.6.24). Let $<\cdot,\cdot>_\delta$ denote the inner product induced on the appropriate neighborhood of $0 \in X_p$ by $M_\delta^*(g)$ then as $\delta \to 0$ this neighborhood encompasses all of X_p and the basis $\mathfrak{h}_p(W'), \mathfrak{h}_p(Z_i)$ is orthonormal relative to the limiting metric. Of course $\mathrm{dVol}_\delta = M_\delta^* \mathrm{dVol}$ tends to the volume form of this metric as well. It is a consequence of (3.6.17) that the metric induced on X_p by this blow–up procedure is the isometric to the Bergman metric on the unit ball.

We define the family of quadratic forms

$$\mathcal{Q}_\delta(f,g) = \int\limits_{X_p} <\overline{\partial} f, \overline{\partial} g>_\delta \mathrm{dVol}_\delta \,. \tag{3.6.27}$$

From the observations in the previous paragraph we conclude that as $\delta \to 0$ the quadratic form $\mathcal{Q}_\delta$ tends to the quadratic form on the unit ball defining the Bergman Laplacian. If we let L_δ denote the second order operator defined by $\mathcal{Q}_\delta$ then the functorial properties of the normal operator construction imply that

$$\lim_{\delta\to 0} L_\delta = N_p(\Delta).$$

This completes the proof of the Proposition.

In addition to the replacement for the tangent bundle, ${}^\Theta T\mho$ we also have a replacement for the cotangent bundle. To define it we simply take the dual bundle to ${}^\Theta T\mho$ which we denote by ${}^\Theta T^*\mho$. Since ${}^\Theta T\mho$ is represented by vector fields that vanish at the boundary, the dual bundle is represented by one forms that blow–up at the boundary. There is locally a basis of the form

$$\frac{d\rho}{\rho}, \frac{\Theta}{\rho}, \frac{\alpha_i}{\rho}, i = 1, \ldots, 2(n-1). \tag{3.6.28}$$

Here the one forms α_i along with Θ restrict to define a coframe for $\partial\mho$. Let $Y_1, \ldots, Y_{2n}$ denote a frame for ${}^\Theta T\mho$ and $\eta_1, \ldots, \eta_{2n}$ denote the dual coframe. If $P \in \mathrm{Diff}^*_m$ then locally P can be expressed in the form

$$P = \sum_{|\alpha|\le m} a_\alpha(q) Y^\alpha. \tag{3.6.29}$$

The Θ–symbol of this operator is the mulitilinear function on the dual bundle given by

$${}^\Theta\sigma^m(P)(\xi\cdot\eta) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha. \tag{3.6.30}$$

Exercise 3.6.31.

(1) Show that the symbol is well defined modulo multilinear functions of degree $m-1$.
(2) Show that the symbol of the Laplace operator of the metric in (3.6.15) is the inner product defined on ${}^\Theta T^*\mho$.

Expressing the Laplace operator of the metric, (3.6.15), in terms of a local coordinate system,

$$(\rho, u, y),\ \partial\mho = \{\rho = 0\},$$

at the $\partial\mho$ we can split the operator up into terms which are homogeneous under the linear action

$$N_\delta(\rho, u, y) = (\delta\rho, u, y). \tag{3.6.32}$$

In terms of this homogeneity structure, the operator has the following structure

$$\Delta = P(u, y, \rho\partial_\rho) + O(\rho). \tag{3.6.33}$$

The first term $P(u, y, x)$ depends in a polynomial fashion on the variable x. The $O(\rho)$ term is a differential operator with the property that it maps smooth functions of the variables (ρ, u, y) which are $O(\rho^k)$ to functions which are $O(\rho^{k+1})$. The leading order term in (3.6.33) can be thought of as a mapping from $\partial\mho$ to operators on $\mathbb{R}_+$, invariant under $\rho \longrightarrow \delta\rho$. This is called the indicial operator; it represents the leading order 'ODE' part of the operator at the boundary. It can be computed by simply 'freezing coefficients'. We denote it by $I_p(\Delta), p \in \partial\mho$.

Exercise 3.6.34.

(1) Prove that the indicial is well defined, that is does not depend on the choice of coordinates. To do this problem you will need to formulate an appropriate notion of equivalence.

(2) For the Laplace operator of the metric, (3.6.15) show that

$$I_p(\mathit{\Delta}) = \frac{1}{4}[(\rho\partial_\rho)^2 - 2n\rho\partial_\rho]. \tag{3.6.35}$$

The indicial operator comes to the fore when we consider the formal solution of boundary value problems

$$(\mathit{\Delta} + s(n-s))u = 0; u \restriction_{\partial\mho} = \rho^\alpha g, \tag{3.6.36}$$

The meaning of the boundary condition is that u has an asymptotic expansion at $\partial\mho$ in powers of ρ beginning with $\rho^\alpha g$. If we assume that u is polyhomogeneous conormal at $\partial\mho$ and has a Taylor expansion of the form

$$u \sim \sum_{j=0}^{\infty} \rho^{\alpha+j} u_j, u_0 = g, \tag{3.6.37}$$

then the indicial operator defines a recursion for the successive terms in the series.

Since u is assumed to satisfy (3.6.38), the leading order equation reads

$$g(p)[I_p(\rho\partial_\rho) + s(n-s)]\rho^\alpha = 0. \tag{3.6.39}$$

Thus we see that the leading order equation determines the exponents α for which the problem (3.6.36) has even a formal solution. In the case at hand

$$\alpha = 2s \text{ or } 2(n-s). \tag{3.6.40}$$

Note the similarity between the technique for formally solving (3.6.36) and the theory of ordinary differential with regular singular points. This is what is meant by saying that the indicial operator is the 'ODE' part of the operator. A main result of the theory we have developed is that problems like (3.6.36) actually have unique, globally defined solutions which are polyhomogeneous conormal distributions. The formal computation above then yields the asymptotic expansion of the true solution at the boundary.

Following the analogy with the model problem considered in the previous section we should now consider a parabolic blowup of the product space: $\mho \times \mho$ in order to resolve the singularities of the resolvent kernel for the Laplace operator. As we saw in the model case this amounts to applying the blow–up procedure to the boundary of the diagonal in the product space that we applied a point at a time above. Since this is a somewhat more abstract situation the procedure we give to define the blow–up will be a bit more formalized.

In outline, we define the blow–up of the zero section in the normal bundle itself. Then, by identifying a neighborhood of the zero section in the normal bundle with a neighborhood of $\partial\Delta$, we transfer the blown–up structure to $\mho^2$. Of course one needs to verify that the definition of the resultant manifold with corners is independent of the identification chosen between the neighborhood of the zero section and the neighborhood of $\partial\Delta$. We will take this for granted in the present lectures.

To define a blow–up of the zero section we need to define a homogeneity structure on the fibers of the normal bundle. The submanifold we want to blow up is $\partial\Delta$ and therefore we need to define a parabolic homogeneity structure on $N_+\partial\Delta$. This is accomplished exactly as in the model case. There are two projections from $\mho^2$ to $\mho$ denoted by π_l and π_r. To that end we set

$$\widetilde{\Theta} = \pi_l^*\Theta - \pi_r\Theta. \tag{3.6.41}$$

Since $\widetilde{\Theta} \restriction_\Delta = 0$, this form is well defined on $N_+\partial\Delta$.

The form $\widetilde{\Theta}$ defines a subbundle of $N_+\partial\Delta$:

$$H = \{(p, v) \in N_+\partial\Delta; \widetilde{\Theta}(v) = 0\}. \tag{3.6.41}$$

As in the model case we need to simply choose a complementing subbundle, which we denote by S. If ρ_{lb} and ρ_{rb} are defining functions for the left and right boundaries of $\mho^2$ then $d\rho_{\mathrm{lb}}$ and $d\rho_{\mathrm{rb}}$ annihilate $T\partial\Delta$. Thus they define one linear forms on $N\partial\Delta$. We require S to lie in the kernel of these forms. A simple calculation in local coordinates shows that this imposes no impediment to constructing S. To be completely rigorous one needs to show that the blown–up space does not depend on the choice of S. In other words if we make a different choice then the resultant spaces are canonically isomorphic.

We have defined a splitting of $N_+\partial\Delta$:

$$N_+\partial\Delta = H \oplus S.$$

Using this splitting we can define a homogeneity structure on the inward pointing normal bundle. Since it is fiber preserving we can define it a fiber at a time. Let $v \in N_{+p}\partial\Delta$ be written as $v = h + s, h \in H_p, s \in S_p$ then

$$M_\delta(v) = \delta h + \delta^2 s. \tag{3.6.42}$$

Using this dilation structure we can define the blowup of the zero section in $N_+\partial\Delta$.

The dilation structure is defined in terms of the one form $\widetilde{\Theta}$. This one form is in turn defined in terms of Θ on $\mho$. Since Θ defines a contact structure on $\partial\mho$ it is a classical theorem of Darboux that we can introduce local coordinates,

$$\rho, x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, u \tag{3.6.43}$$

in a neighborhood of a boundary point, p, such that in these coordinates

$$\Theta = du + \frac{1}{2}\sum_{j=1}^{n-1}(x_i dy_i - y_i dx_i).$$

Notice that this is identical to the model case, see (3.5.11).

We can use two copies of these coordinates on a neighborhood of (p,p) in $\mho^2$. Denote them by

$$(\rho, x, y, u; \rho', x', y', u').$$

By letting

$$\widetilde{x}_i = x_i - x_i', \widetilde{y}_i = y_i - y_i' \text{ and } t = u - u' + \frac{1}{2}\sum[x_j(y_j - y_j') - y_j'(x_j - x_j')],$$

we can obtain an identification between the neighborhood of $(p,p) \in N_+\partial\Delta$ and $(p,p) \in \mho^2$. We simply identify the normal bundle with the span of the coordinate vector fields

$$\partial_{\widetilde{x}_i}, \partial_{\widetilde{y}_i}, \partial_t, \partial_\rho, \partial_{\rho'},$$

in the tangent bundle to $\mho^2$. Then

$$(\rho, \rho', \widetilde{x}, \widetilde{y}, t; x', y', u') \longrightarrow (\rho\partial_\rho + \rho'\partial_{\rho'} + \widetilde{x}\cdot\partial_{\widetilde{x}} + \widetilde{y}\cdot\partial_{\widetilde{y}} + t\partial_t)_{x',y',u'}, \tag{3.6.44}$$

defines such an identification.

From these remarks it is clear that, at least locally, the construction of the blow–up proceeds exactly as in the model case. Since the blow–up is itself a local process, it follows from the independence of choices, assumed above, that the whole construction proceeds exactly as in that case. We introduce local coordinates for the blow–up, lying above the coordinate patch introduced in (3.6.43) by setting

$$\begin{aligned} R^4 &= (r^2 + r'^2 + \frac{1}{2}\sum_{j=1}^{n-1}|z_i - z_i'|^2)^2 + t^2 \\ \rho_{\mathrm{lb}} &= \frac{r}{R}, \rho_{rb} = \frac{r'}{R}, T = \frac{t}{R^2}, X_i = \frac{\widetilde{x}_i}{\sqrt{2}R}, Y_i = \frac{\widetilde{y}_i}{\sqrt{2}R}. \end{aligned} \tag{3.6.45}$$

A comparision shows that this is exactly the coordinate system introduced in the model case in (3.5.17).

We denote the blown–up space by $[\mho^2]_{\partial\Delta,\widetilde{\Theta}}$ with

$$\beta_{\partial\Delta,\widetilde{\Theta}} : [\mho^2]_{\partial\Delta,\widetilde{\Theta}} \longrightarrow \mho^2,$$

the blow down map. As a point set it is defined as

$$[\mho^2]_{\partial\Delta,\widetilde{\Theta}} = \mho^2 \setminus \partial\Delta \amalg N_+\partial\Delta/\mathbb{R}_+, \tag{3.6.46}$$

where the $\mathbb{R}_+$–action is defined in (3.6.42). The $\mathcal{C}^\infty$–structure is defined by using an identification of a neighborhood the zero section in $N_+\partial\Delta$ with a neighborhood of $\partial\Delta$ in $\mho^2$ like (3.6.44) to transfer the dilation structure to a neighborhood of $\partial\Delta$. The ring of smooth functions is defined as the ring generated by pulling back $\mathcal{C}^\infty(\mho^2)$ and the smooth functions in $\mho^2 \setminus \partial\Delta$ homogeneous of degrees 0 and 1 relative to the dilation structure induced on the neighborhood of $\partial\Delta$. In local coordinates these are simply the functions which are smooth as functions of the variables defined in (3.6.45).

As in the model case the blown–up space has three boundary components, a left boundary, a right boundary and a front face, we denote these by $\mathrm{lb}, \mathrm{rb}, \mathrm{ff}$ respectively. We can also define a lift of the diagonal

$$\Delta_\Theta = \overline{\beta^{-1}_{\partial\Delta,\widetilde{\Theta}}(\Delta \setminus \partial\Delta)}.$$

One easily sees in local coordinates that Δ_Θ intersects the front face but is disjoint from lb and rb. This was a primary goal of the construction.

As before the front face is a quarter sphere fibration over the $\partial\Delta$. The fibers again have a canonical structure as lie groups. This arises by lifting sections of $\mathcal{V}_\Theta$ from the left to $\mho^2$. As in the model case a coordinate computation shows that these vector fields lift to $[\mho^2]_{\partial\Delta,\widetilde{\Theta}}$. The lifted vector fields are tangent to the fibers of the front and generate a finite dimensional lie algebra. Working in coordinates it is evident that this precisely the same structure as we obtained by blowing up a point at the boundary. In fact we can define a fiber homomorphism from ${}^\Theta T\mho \restriction_{\partial\mho}$ to the lift of $\mathcal{V}_\Theta$ restricted to the front face in $[\mho^2]_{\partial\Delta,\widetilde{\Theta}}$. This means that for each $p \in \partial\mho$ we can define a homomorphism

$$h_p : {}^\Theta T_p\mho \longrightarrow \beta^*_{\partial\Delta,\widetilde{\Theta}}\pi_l^*(\mathcal{V}_\Theta) \restriction_{\beta^{-1}_{\partial\Delta,\widetilde{\Theta}}(p)} \tag{3.6.47}$$

which depends smoothly on the point p.

As in the previous case we can extend this to elements of Diff^m_Θ. Thus to an operator, P in this ring we associate a family of left invariant operators $N_p(P), p \in \mho$ acting tangent to the fibers of the front face. This is also called the normal operator. It follows from (3.6.47) that this definition agrees with the one given before. We can also consider the behavior of a lifted operator along the left boundary. Clearly it vanishes however the operator $\rho_{\mathrm{lb}}\partial_{\rho_{\mathrm{lb}}}$ is homogeneous of degree zero in an obvious sense, so that the lifted operator has a well defined leading order term. This we call the left indicial operator. At a point $(p, q) \in \mathrm{lb}$ we set this operator equal to $I_p(P)$. A bit of thought is required to see that this makes sense along $\mathrm{lb} \cap \mathrm{ff}$. It is left to the interested reader.

References

The material in this section is taken from the following papers:

EpMe. to appear (1991), *Shrinking Tubes and the $\overline{\partial}$–Neumann Problem.*

EpMeMe. ——— and Gerardo Mendoza, *Resolvent of the Laplacian on strictly pseudoconvex domains*, to appear Acta. Math. (1991).