Exam II, October 18 – Solutions

Problem 1 From Exam I.
Problem 2 From Exam I.
Caution: This is a problem about limits of sequences, not about limits of functions.
Problem 3 Suppose $x_n \to a$ as $n \to \infty$. Prove that $\frac{1}{n}\sum_{k=1}^{n} x_k \to a$ as $n \to \infty$.
Proof: Assume the hypothesis. Choose $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that

$$\left(|x_n - a| < \frac{\varepsilon}{2}\right) \ (\forall n \geq N_1),$$

choose one such. Let $C := \left|\sum_{k=1}^{N_1-1}(x_k - a)\right|$. By the Archimedean property of the real numbers,

$$(\exists N_2 \in \mathbb{N})\left(N_2 > \frac{2C}{\varepsilon}\right),$$

choose one such N_2. Observe that $(N_2 > \frac{2C}{\varepsilon}) \Leftrightarrow (\frac{C}{N_2} < \frac{\varepsilon}{2})$. Set $N := max\{N_1, N_2\}$. Suppose $n \geq N$. Then

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k - a\right| = \left|\frac{1}{n}\sum_{k=1}^{n}(x_k - a)\right| \leq \left|\frac{1}{n}\sum_{k=1}^{N_1-1}(x_k - a)\right| + \left|\frac{1}{n}\sum_{k=N_1}^{n}(x_k - a)\right| \leq$$

$$\leq \frac{C}{n} + \frac{1}{n}\sum_{k=N_1}^{n}|x_k - a| < \frac{C}{n} + \frac{n - N_1 + 1}{n}\frac{\varepsilon}{2}$$

Since $n \geq N = max\{N_1, N_2\}$, we have $\frac{C}{n} \leq \frac{C}{N_2} < \frac{\varepsilon}{2}$ and $\frac{n-N_1+1}{n} \leq 1$. Thus

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k - a\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})$ such that $\left|\frac{1}{n}\sum_{k=1}^{n} x_k - a\right| < \varepsilon$,
QED.
Problem 4 Suppose $\{x_n\}$ is a Cauchý sequence. Suppose $x_n \geq 0$ for infinitely many n and $x_n \leq 0$ for infinitely many n. Prove that $x_n \to 0$ as $n \to \infty$.
Caution: The hypothesis *does not* imply that $(\exists M \in \mathbb{N})$ such that $(x_n \geq 0)(\forall n \geq M)$ or that $(\exists K \in \mathbb{N})$ such that $(x_n \leq 0)(\forall n \geq K)$; that is, the terms of the sequence need not be "eventually nonnegative/nonpositive". For example, take $\{x_n = \frac{(-1)^n}{n}\}$. Actually, if that were true, all terms after $max\{M, K\}$

would be zero.
Proof: Fix $\varepsilon > 0$. Since $\{x_n\}$ is Cauchý, $(\exists N \in \mathbb{N})$ s.t. $(|x_k - x_p| < \varepsilon)(\forall k, p \geq N)$. Let $n \geq N$. Then the element x_n is positive, negative or zero.
Case 1 $x_n \geq 0$. Since $x_m \leq 0$ for infinitely many m, we know $(\exists m \geq N)$ such that $x_m \leq 0$ (else, there would be less than N nonpositive terms in the sequence, which contradicts the assumption). Since $n, m \geq N$, $x_n \leq x_n - x_m = |x_n - x_m| < \varepsilon$.
Case 2 $x_n < 0$. Since $x_m \geq 0$ for infinitely many m, we know $(\exists m \geq N)$ such that $x_m \geq 0$ (else, there would be less than N non-negative terms in the sequence, which contradicts the assumption). But then, since $n, m \geq N$, $x_n < -x_n \leq -x_n + x_m \leq |x_n - x_m| < \varepsilon$. So, in any case, $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})$ such that $(\forall n \geq N)(|x_n| < \varepsilon)$, that is, $\{x_n\} \to 0$.
QED
Notice: We picked $\varepsilon > 0$, then it produced for us N since the sequence is Cauchý; we picked $n \geq N$ and then, considering cases, we deduced $|x_n| < \varepsilon$. In particular, ε, N, n are the same in Case 1 and Case 2. For a different proof see your lab notes.
Problem 5 Let $f(x) = x^3$ for $x \in (-\infty, \infty)$. Show f is differentiable in $(-\infty, \infty)$.
Proof: Let $x_0 \in \mathbb{R}$. We have to prove that the limit

$$\lim_{h \to x_0} \frac{(x_0 + h)^3 - x_0^3}{h}$$

exists. We have

$$\frac{(x_0 + h)^3 - x_0^3}{h} = \frac{x_0^3 + 3x_0^2 h + 3h^2 x_0 + h^3 - x_0^3}{h} = 3x_0^2 + 3hx_0 + h^2$$

From this we obtain

$$\lim_{h \to x_0} \frac{(x_0 + h)^3 - x_0^3}{h} = \lim_{h \to 0}(3x_0^2 + 3hx_0 + h^2) = 3x_0^2 + \lim_{h \to 0}(3hx_0 + h^2).$$

Claim: $\lim_{h \to 0}(3hx_0 + h^2) = 0$
Proof of Claim:
Method 1: Given $\varepsilon > 0$, set $\delta = min\{1, \frac{\varepsilon}{3|x_0|+1}\}$. Then

$$|h| < \delta \Rightarrow \left|3hx_0 + h^2\right| \leq |3x_0 h| + |h^2| < 3|x_0|\delta + \delta^2 \leq 3|x_0|^2\delta + \delta < (3|x_0|^2 + 1)\delta \leq \varepsilon.$$

Thus $|h| < \delta \Rightarrow |3hx_0 + h^2| < \varepsilon$, proving the Claim.
This was the method you were supposed to use in this problem.
Method 2: If you assume that you know properties of limits, and the fact that $\lim_{h \to 0} h = 0$,

$$\lim_{h \to 0}(3hx_0 + h_0^2) = 3x_0 \lim_{h \to 0} h + \lim_{h \to 0} h \lim_{h \to 0} h = 0 + 0 = 0$$

Alternatively, if you assume that you know that polynomials are continuous on $\mathbb{R}$, and if you assume the statement that a function is continuous at an accumulation point of its domain if and only if it has a limit and the limit equals the value at that point, we have

$$\lim_{h\to 0}(3hx_0 + h_0^2) = 3x_0 0 + 0^2 = 0.$$

QED(Claim)
We showed that

$$(\forall x_0 \in \mathbb{R}) \lim_{h\to x_0} \frac{(x_0+h)^3 - x_0^3}{h} = 3x_0^2$$

QED