A-problems

1 Warm-up

1. Let $z, w \in \mathbb{C}$, and $|z| = 1$. Prove that

$$\left|\frac{z-w}{1-z\overline{w}}\right| = 1$$

2. Prove that if *both* $f(z)$ and $\overline{f(z)}$ are analytic in a domain,G, then f is a constant.

2 Some inequalities

1. Let $z, w \in \mathbb{C}$. Prove the inequality

$$||z| - |w|| \leq |z + w| \leq |z| + |w|$$

and interpret it geometrically.

2. Let a be a complex number with $|a| < 1$. Prove that the inequalities

$$|z| \leq 1$$

and

$$\left|\frac{z-a}{1-\overline{a}z}\right| \leq 1$$

are equivalent. Show that equality is achieved for one and the same values of z.

3. Let a and z be complex numbers, satisfying

$$|a| > 1, \left|\frac{z-a}{1-\overline{az}}\right| < 1$$

What can you say about $|z|$?

3 Differentiability and Analyticity

1. Let f be *analytic* in some domain, G, and suppose that all of its values are contained in one of the following curves:
   a) $ax + by + c = 0, a, b, c - const.$
   b) $(x - a)^2 + (y - b)^2 = R^2,\ a, b, R - const., R > 0$
   Prove that f is a constant function.

   Non-Example: All of the values of $f(z) = |z|^2$ are contained in the positive real line, and it is nonconstant.But it is nowhere analytic.

   Hint: If $f = u + iv$, the functions u, v should satisfy the corresponding equation, e.g., $au(x, y) + bv(x, y) + c = 0$ for a). Differentiate it wrt x and y and apply the Cauchy-Riemann equations.

   Note: In general, it can be shown that the values of a non-constant analytic function cannot be contained in any curve, and that it transforms open sets to open sets (i.e., the image of a domain should be a domain).

2. Show that the function $u : \mathbb{R}^2 \to \mathbb{R}$, $u(x, y) = \sqrt{|xy|}$ has partial derivatives at $(0, 0)$ but is not differentiable there.

3. Prove that the function defined by:

$$f(z) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} + i\frac{xy}{\sqrt{x^2+y^2}}, & z = x + iy \neq 0 \\ 0 & z = 0 \end{cases}$$

   satisfies the Cauchy-Riemann equations at $z = 0$ but has no derivative at $z = 0$. Why can something like this happen?

4 And something else ...

1. Let γ be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Compute the integral $\int_\gamma \frac{dz}{z}$ in two different ways and prove the identity:

$$\int_0^{2\pi} \frac{dt}{a^2cos^2t + b^2sin^2t} = \frac{2\pi}{ab}$$

2. Suppose the function f is continuous in the domain $|z - z_0| > r_0$. Let C_r be the circle with radius r, $C_r : |z - z_0| = r > r_0$. Let $M(r) = max|f(r)|$ on C_r, and suppose $rM(r) \to 0$ as $r \to \infty$. Show that $\int_{C_r} f(z)dz \to 0$, as $r \to \infty$.