A-problems
Problem Set 2

Solve **two** problems from the section 'Series' and **ten** integrals from the second section. The Fresnel integrals – problem 3 – are one integral.

1 Series

1. Consider the power series
$$\sum_{n=0}^{\infty} a_n z^n$$
and suppose it has radius of convergence R. Show that if we differentiate it term by term, the new power series still has radius of convergence R.

2. Write the Laurent expansion of $\frac{1}{z-k}$ in the domain $|z| > |k|, k \in \mathbb{R}, k^2 < 1$, and by setting $z = e^{i\theta}$ prove the identities:
$$\sum_{n=1}^{\infty} k^n \sin(n\theta) = \frac{k \sin\theta}{1 + k^2 - 2k\cos\theta}$$
$$\sum_{n=1}^{\infty} k^n \cos(n\theta) = \frac{k \cos\theta - k^2}{1 + k^2 - 2k\cos\theta}$$
Hint: Don't get petrified by the expressions. It's easier than you think.

3. Find the Laurent expansion of $\cos\frac{z}{1-z}$ in a neighbourhood of $z = 1$ and in $|z| > 1$.
Hint: Before starting, use some trigonometry.

4. What is the difference in the behaviour of the functions
$$y = \begin{cases} \exp\{-\frac{1}{x^2}\}, x \neq 0, x \in \mathbb{R} \\ 0, x = 0 \end{cases}$$
and
$$w = \begin{cases} \exp\{-\frac{1}{z^2}\}, z \neq 0, z \in \mathbb{C} \\ 0, z = 0 \end{cases}$$

in a neighbourhood of $x = 0$, resp. $z = 0$?
The question is somewhat philosphical, but the answer is not. The difference is very concrete. In some sense one of these functions is very 'good' while the other is very 'bad'. Which is which and what does this mean?

2 Computing integrals with the Residue Theorem

1. Compute the following integrals:

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2(x^2+b^2)},\ a > 0,\ b > 0$$

$$\int_0^\infty \frac{dx}{(1+x^2)^n},\ n > 0, n \in \mathbb{Z}$$

$$\int_{-\infty}^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx,\ (a > b > 0)$$

$$\int_0^{2\pi} \frac{d\theta}{(p+q\cos\theta)^2},\ p > q > 0$$

2. Compute the integral

$$\int_0^\infty \frac{dx}{1+x^n},\ n > 1, n \in \mathbb{Z}$$

Hint: Integrate the function $\frac{1}{1+z^n}$ along the closed contour, consisting of the segment $[0, R]$ along the *Re* axis, followed by the arc $z = Re^{it},\ 0 \leq t \leq \frac{2\pi}{n}$, followed by the line segment $z = re^{\frac{2\pi i}{n}},\ r \in [0, R]$.

3. In this problem we compute the *Fresnel integrals*

$$\int_0^\infty \sin(x^2)dx,\ \int_0^\infty \cos(x^2)dx$$

They are very important in wave/fibre optics, lasers,difraction and interference phenomena, etc. For that, apply Cauchý's Theorem for $\int_C e^{iz^2}dz$, where C is the contour from the previous problem, with $n = 8$.

4. Compute the (Cauchý principal value of the) improper integrals:

$$\int_{-\infty}^{\infty} \frac{(x^2+2)dx}{x^5-x^4+x^3-x^2+x-1}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x+a)^2+b^2} dx, \ a>0, \ b>0$$

$$\int_{0}^{\infty} \frac{x \ \sin \ mx}{x^2+b^2} dx, m>0, \ b>0$$

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x(x^2+b^2)} dx, a>0, b>0$$

$$\int_{0}^{\infty} \frac{\cos \alpha x - \cos \beta x}{x^2} dx, \alpha \geq 0, \beta \geq 0$$

Hint: Do not split that integral in two integrals.

5. Compute the integral

$$\int_{0}^{\infty} \frac{\cos \ x - e^{-x}}{x} dx$$

Hint: Integrate the function $\frac{e^{iz}-e^z}{z}$ along the contour consisting of the segments $[r,R]$ and $[ir,iR]$ on the Re and Im axis, respectively, and the two arcs re^{it},Re^{it}, $t \in [0,\pi/2]$; Here the segments are not given in order, and the contour is oriented, as usual, positively. Then take the limit $r \to 0$, $R \to \infty$.

6. Compute the integrals

$$\int_{0}^{\pi} \frac{\cos 2\theta}{1+k^2-2k\cos\theta} d\theta, \ k^2<1$$

$$\int_{0}^{\pi} \sin^{2n}\theta d\theta$$