A-problems
Problem Set 2 - some answers

Solve **two** problems from the section 'Series' and **ten** integrals from the second section. The Fresnel integrals – problem 3 – are one integral.

1 Series

1. Consider the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

and suppose it has radius of convergence R. Show that if we differentiate it term by term, the new power series still has radius of convergence R. Apply the ratio test to the differentiated series, and use that

$$\lim \left| \frac{n a_n}{(n-1)a_{n-1}} \right| = \lim \frac{n}{n-1} \lim \left| \frac{a_n}{a_{n-1}} \right| = R$$

2. Write the Laurent expansion of $\frac{1}{z-k}$ in the domain $|z| > |k|, k \in \mathbb{R}, k^2 < 1$, and by setting $z = e^{i\theta}$ prove the identities:

$$\sum_{n=1}^{\infty} k^n \sin(n\theta) = \frac{k \sin\theta}{1 + k^2 - 2k\cos\theta}$$

$$\sum_{n=1}^{\infty} k^n \cos(n\theta) = \frac{k\cos\theta - k^2}{1 + k^2 - 2k\cos\theta}$$

Hint: Don't get petrified by the expressions. It's easier than you think.

Just follow what the problem suggests:

$$\frac{1}{z-k} = \frac{1}{z}\frac{1}{1-k/z} = \frac{1}{z}\sum_{0}^{\infty}\frac{k^n}{z^n} =$$

$$= \sum_{0}^{\infty}\frac{k^n}{z^{n+1}} = \sum_{0}^{\infty} k^n e^{-i(n+1)\theta}$$

Now we just have to compare the real and imaginary part of both sides. Multiplying LHS by $\frac{\bar{z}-k}{\bar{z}-k}$ (remember, $k \in \mathbb{R}$), we get

$$LHS = \frac{\cos\theta - k}{1 + k^2 - 2k\cos\theta} - i\frac{\sin\theta}{1 + k^2 - 2k\cos\theta}$$

$$RHS = \sum_0^\infty k^n \cos(n+1)\theta - i\sin(n+1)\theta$$

Multiplying both sides by k and setting Re and Im parts equal gives the result.

3. Find the Laurent expansion of $\cos\frac{z}{1-z}$ in a neighbourhood of $z_0 = 1$ and in $|z| > 1$.
   Hint: Before starting, use some trigonometry.
   Expanding around $z_0 = 1$, we need powers of $(z-1)$, so write $\frac{z}{1-z} = -1 + \frac{1}{1-z}$. Then $\cos\frac{z}{1-z} = \cos 1 \cos\frac{1}{1-z} - \sin 1 \sin\frac{1}{1-z}$ and we use the MacLaurin expansions of sin and cos.

4. What is the difference in the behaviour of the functions

$$y = \begin{cases} \exp\{-\frac{1}{x^2}\}, x \neq 0, x \in \mathbb{R} \\ 0, x = 0 \end{cases}$$

and

$$w = \begin{cases} \exp\{-\frac{1}{z^2}\}, z \neq 0, z \in \mathbb{C} \\ 0, z = 0 \end{cases}$$

in a neighbourhood of $x = 0$, resp. $z = 0$?
The question is somewhat philosphical, but the answer is not. The difference is very concrete. In some sense one of these functions is very 'good' while the other is very 'bad'. Which is which and what does this mean?

The first function is infintely differentiable everywhere – has continous derivatives of any order at any point. Of course, the only point where this needs a proof is $x = 0$. The second function is not analytic at $z = 0$ (and doesn't have a derivative there). In fact, it has an essential singularity there.

2 Computing integrals with the Residue Theorem

Most of the integrals that follow take one page to write down, so for some problems I am giving only the answers or a sketch of a solution.

1. Compute the following integrals:

(a) The integral

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$$

Answer: $\frac{\pi}{6}$.

Let $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$. We have Res $(f,i) = i/6$, Res$(f,2i) = -i/3$, and the integral is given by

$$\frac{1}{2}2\pi i\,(\text{ Res }(f,i) + \text{ Res }(f,2i)) = \frac{\pi}{6}$$

(b) The integral

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2(x^2+b^2)},\ a>0,\ b>0$$

Answer:

$$\frac{(2a+b)\pi}{4a^3b(b+a)^2}.$$

Let $f(z) = \frac{1}{(z^2+a^2)^2(z^2+b^2)}$. The value of the integral is

$$2\pi i\,(\text{ Res }(f,ia) + \text{ Res }(f,ib))$$

$$\text{Res }(f,ia) = \frac{1}{4i}\frac{b^2-3a^2}{a^3(b^2-a^2)^2},\ \text{ Res }(f,ib) = \frac{1}{2ib(a^2-b^2)^2}$$

(c) The integral

$$\int_0^\infty \frac{dx}{(1+x^2)^n},\ n>0, n\in\mathbb{Z}$$

Answer:

$$\pi\frac{(2n-2)!}{[(n-1)!]^2}\frac{1}{2^{2n-1}}$$

To evaluate the integral, one sets $f(z) = \frac{1}{(1+z^2)^n} = \frac{1}{(z-i)^n(z+i)^n}$. Then the integral is given by

$$\frac{1}{2}2\pi i\text{ Res }(f,i),\ \text{and}\ \text{ Res }(f,i) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\frac{1}{(z+i)^n}\bigg|_{z=i}$$

The result is obtained after one shows that

$$((z+i)^{-n})^{(n-1)}(i) = -\frac{(2n-2)!}{(n-1)!}\frac{i}{2^{2n-1}}$$

(d) The integral

$$\int_{-\infty}^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx,\ (a>b>0)$$

Answer:
$$\frac{\pi}{a^2-b^2}\left(-\frac{e^{-a}}{a}+\frac{e^{-b}}{b}\right)$$

Consider $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$. The integral has value

$$\text{Re }\{2\pi i(\text{ Res }(f, ia) + \text{ Res }(f, ib))\}$$

$$\text{Res }(f, ia) = i\frac{e^{-a}}{2a(a^2-b^2)},\ \text{Res }(f, ib) = i\frac{e^{-b}}{2b(b^2-a^2)}$$

(e) The integral
$$\int_0^{2\pi} \frac{d\theta}{(p+q\cos\theta)^2},\ p > q > 0$$

Answer:
$$\frac{2\pi p}{(p^2-q^2)^{3/2}}$$

We convert this integral to an integral over the unit circle, $C : |z| = 1$,

$$\int_0^{2\pi} \frac{d\theta}{(p+q\cos\theta)^2} = \oint_C \frac{\frac{dz}{iz}}{(p+\frac{q}{2}(z+\frac{1}{z}))^2} =$$

$$= \frac{1}{i}\oint_C \frac{z\ dz}{(zp+\frac{z^2 q}{2}+\frac{q}{2})^2} = \frac{4}{iq^2}\oint \frac{z\ dz}{(z^2+2\frac{p}{q}z+1)^2}$$

Look at the quadratic equation in the denominator:

$$z^2+2\frac{p}{q}z+1 = (z-\alpha_1)(z-\alpha_2),\ \alpha_{1,2} = -\frac{p}{q} \pm \sqrt{\frac{p^2}{q^2}-1}$$

As $p > q > 0$, $p/q > 1$, so $-\frac{p}{q} < -1$, and hence $\alpha_1 = -\frac{p}{q} - \sqrt{\frac{p^2}{q^2}-1}$ is *always* outside C.
Is α_2 inside C? Suppose not. This can happen in two cases: either $\alpha_2 < -1$ or $\alpha_2 > 1$. For convenience, let $\kappa = \frac{p}{q}$, so $\alpha_2 = -\kappa + \sqrt{\kappa^2-1}, \kappa > 1$. Then:

$$\alpha_2 < -1 \Leftrightarrow -\kappa + \sqrt{\kappa^2-1} < -1 \Leftrightarrow \kappa - 1 > \sqrt{\kappa^2-1}$$
$$\Leftrightarrow (\kappa-1)^2 > \kappa^2 - 1 \Leftrightarrow 2\kappa < 2 \Leftrightarrow \kappa < 1, \text{contradiction}$$

Similarly,

$$\alpha_2 > 1 \Leftrightarrow -\kappa + \sqrt{\kappa^2-1} > 1 \Leftrightarrow \kappa + 1 < \sqrt{\kappa^2-1}$$

$$\Leftrightarrow (\kappa+1)^2 < \kappa^2 - 1 \Leftrightarrow 2\kappa + 1 < -1 \Leftrightarrow \kappa < -1,\ \text{contradiction}$$

So, we now know that $\alpha_2 = -\frac{p}{q} + \sqrt{\frac{p^2}{q^2} - 1}$ is *always* inside C.
The integral has value

$$2\pi i \text{ Res } \left(\frac{4}{iq^2} \frac{z}{(z^2 + 2\frac{p}{q}z + 1)^2},\ \alpha_2 \right) = \frac{2\pi p}{(p^2 - q^2)^{3/2}}$$

2. Compute the integral

$$\int_0^\infty \frac{dx}{1 + x^n},\ n > 1, n \in \mathbb{Z}$$

Hint: Integrate the function $\frac{1}{1+z^n}$ along the closed contour, consisting of the segment $[0, R]$ along the Re axis, followed by the arc $z = Re^{it},\ 0 \leq t \leq \frac{2\pi}{n}$, followed by the line segment $z = re^{\frac{2\pi i}{n}},\ r \in [0, R]$.
Comment: You can do the case n even without the hint, but rather by computing half of the same integral on $(-\infty, \infty)$, and taking into account the resifues at those $n-$th roots of -1 which are in the upper half plane. But this is more time-consuming, and works only for the case of even n.
Answer: $\frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}$

So, the poles of the function $\frac{1}{1+z^n}$ are at the roots of -1, the 'first' one being $\xi_0 = \exp\{\frac{i\pi}{n}\}$, and the second $\exp\{\frac{i3\pi}{n}\}$, so the curve in the hint surrounds only one pole. Let's denote by I the integral we want to compute. The part of the contour which is on the real axis gives us I, the one on the segment $z = re^{\frac{2\pi i}{n}},\ r \in [0, R]$ gives

$$\int \frac{dz}{1 + z^n} = \int e^{\frac{2\pi i}{n}} \frac{dr}{1 + r^n} = e^{\frac{2\pi i}{n}} I$$

Finally, the integral on the arc vanishes when we take $R \to \infty$, because is $z = Re^{it}$, $|1 + z^n| \geq |1 - |z|^n| = |R^n - 1|$, hence

$$\left| \int_{C_R} \frac{dz}{1 + z^n} \right| \leq \frac{2\pi R}{n} \frac{1}{|R^n - 1|} \to 0, \text{as} R \to \infty$$

So we have

$$I + 0 - e^{\frac{2\pi i}{n}} I = Res(f, e^{\frac{\pi i}{n}})$$

The shortest way to compute the residue is using L'Hopital, as

$$\lim_{z \to \xi_0} \frac{z - \xi_0}{1 + z^n} = \lim_{z \to \xi_0} \frac{1}{nz^{n-1}} = \frac{1}{n}\xi_0^{-n+1} = -\frac{1}{n}\xi_0 = -e^{\frac{i\pi}{n}} \frac{1}{n}$$

Now we just put back everything:

$$I(1 - e^{\frac{2\pi i}{n}}) = -\frac{2\pi i}{n} e^{\frac{i\pi}{n}}$$

$$I = -\frac{\pi}{n} 2i \frac{e^{\frac{i\pi}{n}}}{1 - e^{\frac{2\pi i}{n}}} = \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

3. In this problem we compute the *Fresnel integrals*

$$\int_0^\infty \sin(x^2)dx, \int_0^\infty \cos(x^2)dx$$

They are very important in wave/fibre optics, lasers,difraction and interference phenomena, etc. For that, apply Cauchý's Theorem for $\int_C e^{iz^2}dz$, where C is the contour from the previous problem, with $n = 8$.

This is very similar to the above problem, but there are no singular points inside the contour, and the integral on the line segment reduces to the Gauß integral. The **answer** is $\sqrt{\frac{\pi}{8}}$ in both cases.

4. Compute the (Cauchý principal value of the) improper integrals:

(a)
$$\int_{-\infty}^\infty \frac{(x^2+2)dx}{x^5 - x^4 + x^3 - x^2 + x - 1}$$

Answer: $-\pi\sqrt{3}$

Notice:

$$z^5 - z^4 + z^3 - z^2 + z - 1 = (z-1)(z^4+z^2+1) = (z-1)(z-z_0)\ldots(z-z_3),$$

where $z_0 = e^{\frac{\pi i}{3}}, z_1 = e^{\frac{4\pi i}{3}}, z_2 = e^{-\frac{\pi i}{3}}, z_3 = e^{\frac{2\pi i}{3}}$. Let $f = \frac{z^2+2}{z^5-z^4+z^3-z^2+z-1}$. The integral is given by

$$\pi i \text{ Res}(f, 0) + 2\pi i \text{ Res}(f, z_0) + 2\pi i \text{ Res}(f, z_3)$$

We have

$$\text{Res}(f, 1) = 1, \text{ Res}(f, z_0) = \frac{-1 + i\sqrt{3}}{4}, \text{ Res}(f, z_3) = \frac{1}{4\sqrt{3}}(-\sqrt{3} + i).$$

(b)
$$\int_{-\infty}^\infty \frac{\cos x}{(x+a)^2 + b^2}dx, \; a > 0, \; b > 0$$

Answer: $\frac{\pi e^{-b}}{b} \cos a$

Notice: $(z + a)^2 + b^2 = (z + a + ib)(z + a - ib) = (z - z_0)(z - z_1)$, where $z_1 = -a + ib, z_0 = \overline{z_1}$.
Let $f = \frac{e^{iz}}{(z+a)^2+b^2}$. The integral is given by Re $(2\pi i$ Res $(f, z_1))$, and $2\pi i$Res $(f, z_1) = 2\pi i \frac{e^{iz_1}}{z_1 - z_0} = \frac{\pi e^{-b}}{b}(\cos a - i \sin a)$.

(c)

$$\int_0^\infty \frac{x\ \sin\ mx}{x^2+b^2}dx, m>0,\ b>0$$

Answer: $\frac{\pi}{2}e^{-bm}$

Let $f(z)=\frac{ze^{izm}}{z^2+b^2}$. The integral is given by

$$\frac{1}{2}\ \text{Im}\ (2\pi i\ \text{Res}\ (f,ib)) = \ \text{Im}\ \frac{i\pi e^{-bm}}{2} = \frac{\pi}{2}e^{-bm}$$

(d)

$$\int_{-\infty}^\infty \frac{\sin ax}{x(x^2+b^2)}dx, a>0, b>0$$

Answer: $\frac{\pi}{b^2}(1-e^{-ab})$

Let $f(z)=\frac{e^{iza}}{z(z^2+b^2)}$. Then the integral is

$$\text{Im}\ (\pi i\ \text{Res}\ (f,0){+}2\pi i\ \text{Res}\ (f,ib)) = \ \text{Im}\ \left(\frac{\pi i}{b^2}+\frac{2\pi i e^{-ab}}{(ib)(2ib)}\right) = \frac{\pi}{b^2}(1{-}e^{-ab})$$

(e)

$$\int_0^\infty \frac{\cos\alpha x-\cos\beta x}{x^2}\ dx, \alpha\geq 0, \beta\geq 0$$

Hint: Do not split that integral in two integrals.
Answer: $\frac{\pi}{2}(\beta-\alpha)$.

This one is interesting. You may suspect that you'll get a 2-nd order pole at $z=0$, but it's not the case. First, just to get going, notice that the real function $\frac{\cos\alpha x-\cos\beta x}{x^2}$ has a limit at zero (which is actually $\frac{\alpha^2-\beta^2}{2}$), rather than diverging.
Nevermind that, we compute the integral using the usual contour for these problems ($C_R=[-R,R]\cup Re^{i\theta}, \theta\in[0,\pi]$), so we have

$$\frac{1}{2}Re \lim_{R\to\infty}\oint_{C_R}\frac{e^{i\alpha z}-e^{i\beta z}}{z^2} = \frac{1}{2}Re\{\pi i Res(0)\}$$

We have $\frac{1}{2}$ because the function is even, and πi instead of $2\pi i$ because we have a simple pole on the real axis. To see that it's a simple pole and to compute the residue we just look at the series:

$$\frac{e^{i\alpha z}-e^{i\beta z}}{z^2} = \frac{1-i\alpha z+\ldots-1+i\beta z+\ldots}{z^2} = \frac{i(\alpha-\beta)}{z}+\ldots$$

So the residue is $i(\alpha-\beta)$.
If you split it in two, you'll get two functions with second orger poles on the real axis.

5. Compute the integral

$$\int_0^\infty \frac{\cos\ x - e^{-x}}{x} dx$$

Hint: Integrate the function $\frac{e^{iz}-e^z}{z}$ along the contour consisting of the segments $[r, R]$ and $[ir, iR]$ on the Re and Im axis, respectively, and the two arcs re^{it},Re^{it}, $t \in [0, \pi/2]$; Here the segments are not given in order, and the contour is oriented, as usual, positively. Then take the limit $r \to 0$, $R \to \infty$.
Answer: 0.

6. Compute the following integrals

   (a) The integral

   $$\int_0^\pi \frac{\cos 2\theta}{1 + k^2 - 2k \cos\theta} d\theta,\ k^2 < 1$$

   Answer: $\frac{\pi k}{1-k^2}$

   (b) The integral

   $$\int_0^\pi \sin^{2n}\theta d\theta$$

   Answer: $\pi\frac{(2n)!}{(2^n n!)^2}$.
   We notice that this is $\frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta d\theta$. We convert the latter integral into an integral over the unit circle, $C : |z| = 1$:

$$\frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta d\theta = \frac{1}{(2i)^{2n+1}} \oint_C \frac{(z^2-1)^{2n}}{z^{2n+1}}\ dz =$$

$$= \frac{1}{(2i)^{2n+1}} 2\pi i \text{ Res } \left(\frac{(z^2-1)^{2n}}{z^{2n+1}}, 0\right) = \frac{1}{(2i)^{2n+1}} 2\pi i \lim \frac{1}{n!}\frac{d^n}{dz^n}(z^2-1)^{2n} =$$

$$= \frac{1}{(2i)^{2n+1}} 2\pi i \frac{(-1)^n (2n)!}{(n!)^2} = \pi\frac{(2n)!}{(2^n n!)^2}$$