

Cluster Points

Let $\{x_n\}$ be a sequence of real numbers. Consider the statements:

- A. $(\forall \epsilon > 0)(\forall M \in \mathbb{N})(\exists m \geq M)$ s.t. $|x_m - c| < \epsilon$
- B. $(\forall \epsilon > 0)(\exists$ infinitely many $m \in \mathbb{N})$ s.t. $|x_m - c| < \epsilon$
- C. $(\forall \epsilon > 0)(\exists m \in \mathbb{N}, m \neq c)$ s.t. $(|x_m - c| < \epsilon)$

We have

$$A. \Leftrightarrow B.$$

and this is just the definition of cluster point. Now, if $(x_n \neq c)(\forall n \in \mathbb{N})$ we have moreover

$$A. \Leftrightarrow B. \Leftrightarrow C.$$

To prove that, use the fact that

$$\frac{\epsilon}{1 + \epsilon} \frac{|x_n - c|}{1 + |x_n - c|} < \min \{\epsilon, |x_n - c|\}$$

If there are some n for which $x_n = c$, you have only

$$C. \Rightarrow A. \Leftrightarrow B.$$