Math 241, Exam 1

May 29, 2007

Name: ______________________________

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In order to receive full credit you need to show all your work .

Score		
1	4	
2	4	
3	4	
4	4	
5	4	
Total	20	

Convenient formulas and notation:

- We can write $z \in \mathbb{C}$ as

$$z = x + iy = |z|(\cos\theta + i\sin\theta) = re^{i\theta},\ r = |z| = \sqrt{x^2 + y^2}$$

$$x = Re(z) = \frac{z + \bar{z}}{2},\ y = Im(z) = \frac{z - \bar{z}}{2i},\ \bar{z} := x - iy$$

- Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- The Cauchý-Riemann equations for $f = u + iv$ are

$$u_x = v_y,\ u_y = -v_x$$

1. Find and draw all $z \in \mathbb{C}$, such that

$$z^3 + i = 0$$

Draw the unit circle on the same picture. Make the picture **big**.

Hint: **Do not** quote from memory de Moivre's formula, but just write what does the equation say for $z = re^{i\theta}$.

This was a practice problem: Problem 3, Section 3 , p.2.

Following the hint,if $z = re^{i\theta}$, $z^3 = r^3 e^{i3\theta} = -i = e^{-i\frac{\pi}{2}}$.
Taking the modulus of both sides gives
$|z^3| = r^3 = |-i| = 1$, i.e., $r^3 = 1$. But $r \geq 0 \Rightarrow r = 1$. So

$$z^3 = e^{i3\theta} = e^{-i\frac{\pi}{2}}$$
$$3\theta = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$$
$$\theta = \frac{-\frac{\pi}{2} + 2n\pi}{3} = (4n-1)\frac{\pi}{6}, n \in \mathbb{Z}$$

But there will be only 3 distinct values, and we can get these by taking $n = 0, 1, 2$.
Comment:
We know that only 3 values are distinct from the 'theory' in the textbook (there are k distinct $k-$th roots) or by noticing that $4(n+3)-1 = 4n-1+12$, so $(4(n+3)-1)\frac{\pi}{6} = (4n-1)\frac{\pi}{6} + 2\pi$.
The roots are thus

$$e^{-i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ e^{i\frac{7\pi}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i, e^{i\frac{pi}{2}} = i$$

The lie on the unit circle, as the vertices of an equilateral triangle, and one of the vertices is i.

2. Find a function u such that $f = u + iv$ is analytic in $\mathbb{C}$ and

$$v(x, y) = x^2 - y^2$$

This is either a problem from the book or very close to one such (I am relying on my memory here).

If we write the Cauchý-Riemann equations for u and v, we get

$$u_x = v_y = -2y, u_y = -v_x = -2x$$

Integrating formally the first equation with respect to x we get $u = -2yx + f(y) \Rightarrow u_y = -2x + f'(y)$. Comparting with the second Cauchý-Riemann equation, we get $-2x = -2x + f'$, $f' = 0$, $f = \mathit{const} =: C$. So I can take $u = -2yx$ (or $u = -2yx + C$; the problem says "find **a** function u"). So $f = -2yx + i(x^2 - y^2)$

Comment: Here we just found two functions, u and v, satisfying the Cauchý-Riemann equations. They are actually polynomials, so they are $\mathbb{R}$-differentiable and thus analyticity is guaranteed.

3. Consider the set S defined by

$$z\overline{z} + 2iz - 2i\overline{z} + 1 = 0$$

Is it open/closed/both/neither? Draw a picture. Give the equation of an open disk with centre $2i$ not containing any points of S.

The equation giving this set is a special case of Practice Problems, p.3, Section 4, Problem 5 $(l = 2, m = 1)$.

We have

$$z\overline{z} + 2i(z - \overline{z}) + 1 = 0$$

If $z = x + iy$, this becomes

$$x^2 + y^2 + 2i(2iy) + 1 = 0$$

$$x^2 + y^2 - 4y + 1 = 0$$

Completing the square, we get $x^2 + (y - 2)^2 - 3 = 0$, so

$$x^2 + (y - 2)^2 = 3$$

So S is a circle with centre $2i$ and radius $\sqrt{3}$. We can also write the same equation as $|z - 2i| = \sqrt{3}$.
An open disk with centre $2i$, not containing any points of S is, for instance, $|z - 2i| < 1$, or $|z - 2i| < \sqrt{3}$; in general, any disk $|z - 2i| < R, R \leq \sqrt{3}$ will do.
The set S is a closed subset of $\mathbb{C}$. This is clear from the picture, but if you have to write it properly, here is one way. We show that the complement of S is open. Why is this true? Take any z_0 **not** in S. This means $|z_0 - 2i| \neq \sqrt{3}$, but it is equal to something, call that something δ, so $|z_0 - 2i| = \delta \neq \sqrt{3}$. Then $|z - z_0| < |\delta - \sqrt{3}|$ is an open disk around z_0, not containing any points of S.
NB Here I needed to take $|\delta - \sqrt{3}|$ because I don't know which one is bigger, i.e., I don't know if z_0 is inside or outside the circle S.

4. Consider the function

$$f(z) = x^2 - x + y + i(y^2 - 5y - x)$$

Determine the points at which it is
1) $\mathbb{C}-$ differentiable
2) analytic

This problem is from the book.
Denote $u = x^2 - x + y$, $v = y^2 - 5y - x$. These are $\mathbb{R}$-differentiable functions (because they are polynomials, and are continuous with continuous partials everywhere). Their partial derivatives are $u_x = 2x - 1, u_y = 1, v_x = -1, v_y = 2y - 5$. The Cauchý-Riemann equations are then $2x - 1 = 2y - 5$, $-1 = -1$, so this means $y = x + 2$ which gives us the equation of a line.
Thus f is $\mathbb{C}$-differentiable at all points on the line $y = x + 2$, i.e., at all $z = x + i(x + 2), x \in \mathbb{R}$. Since the line $y = x + 2$ is a closed subset of $\mathbb{C}$, f is nowhere analytic.

5. Can there be $z \in \mathbb{C}$ such that $e^z = -1$? If no, why? If yes, find all such z.

We just write the definition of e^z, $z = x + iy$. That is, $e^z := e^x e^{iy} = -1$. If we take the absolute value of both sides, we get $e^x = 1 \Rightarrow x = 0$. Thus we want $e^{iy} = -1 = e^{i\pi}$. But this means that $y = \pi + 2n\pi = (2n+1)\pi, n \in \mathbb{Z}$. So any z of the form $z = e^{i(2n+1)\pi}$ will satisfy $e^z = -1$ and these are all such. Notice that, of course, none of these is a real number, there are no real z with this property.