Math 241, Exam 4

June 19, 2007

Name: ____

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In order to receive full credit you need to show all your work .

Score		
1	4	
2	4	
3	4	
4	4	
5	4	
Total	20	

1. Compute the integral

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

Make sure you indicate clearly all the contours of integration and all the uses of the residue theorem,etc.
We use the usual for these problems contour, $C_R = [-R, R] \cup \{Re^{i\theta}, \theta \in [0, \pi]\}$, and take $R \to \infty$.

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$= \frac{1}{2} 2\pi i Res(\frac{1}{(z^2+1)^2}, i) = \frac{1}{2}(2\pi i)(-\frac{i}{4}) = \frac{\pi}{4}$$

Here I used that $\frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}$, and

$$Res(\frac{1}{(z^2+1)^2}, i) = \lim_{z\to i} \frac{d}{dz}\frac{1}{(z+i)^2} = \frac{-2}{(2i)^3} = -\frac{i}{4}$$

2. Compute Cauchy principal value of the integral

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{dx}{x^3 + 2x^2 + 2x}$$

Make sure you indicate clearly all the contours of integration and all the uses of the residue theorem,etc.

As $\frac{1}{x^3+2x^2+2x} = \frac{1}{x(x^2+2x+2)}$, we are using the contour from Problem 1, but with an indentation around $z_0 = 0$.

Notice that

$$\frac{1}{z^3 + 2z^2 + 2z} = \frac{1}{z(z + 1 - i)(z + 1 + i)} =: f(z)$$

So the integral above is equal to

$$\pi i Res(f, 0) + 2\pi i Res(f, -1 + i) = \frac{\pi i}{2} + \frac{\pi}{2}(-1 - i) = -\frac{\pi}{2}$$

3. Find the Fourier expansion of the function $f(x) = x, \ x \in [-\pi, \pi]$.
After computing the coefficients, write out the series you get.
The function is **odd**, so the expansion will contain only $\sin nx$, with coefficients

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx dx = -\frac{2}{n\pi}\int_0^\pi x d(\cos nx) = -\frac{2}{n\pi} x \cos nx\Big|_0^\pi + \frac{2}{n\pi}\int_0^\pi \cos nx dx =$$

$$= \frac{2(-1)^{n+1}}{n}$$

So the Fourier expansion is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \ldots$$

If you look at it, already the first term is not such a bad approximation...

4. Find the Fourier exapansion of $f(x) = \sin^2 x,\ x \in [-\pi, \pi]$.
After computing the coefficients, write out the series you get (and the first two nonzero terms).

The function f is **even**, so there will be only cosines in its expansion.

Recall: The set $\{1, \cos nx, \sin nx\}$ is orthogonal on $[-\pi, \pi]$ by a very improtant homework problem.

Method 1: you see it...

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$$

That's it – it's already expanded, $\frac{a_0}{2} = \frac{1}{2}$ (so $a_0 = 1$), $a_2 = -\frac{1}{2}$, and $a_k = 0, k \neq 0, 2, b_k = 0$.

Method 2: you see it later...

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x \cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\frac{1}{2} - \frac{1}{2}\cos 2x)\cos nx dx =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos nx dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos 2x \cos nx dx$$

But both of these integrals are inner products – this is

$$\frac{1}{2\pi}(1, \cos nx) - \frac{1}{2\pi}(\cos 2x, \cos nx)$$

By the orthogonality of the trigonometric system (above!), these integrals are zero, unless $n = 0$ or $n = 2$, in which cases they give

$$\text{if } \ n = 0, a_0 = \frac{1}{2\pi}\|1\|^2 = \frac{1}{2\pi}2\pi = 1$$

$$\text{if } \ n = 2, a_2 = -\frac{1}{2\pi}\|\cos 2x\|^2 = -\frac{1}{2\pi}\pi = -\frac{1}{2}$$

Method 3: you don't see it

This methods amounts to redoing a piece of the above-mentioned homework problem: you start as in Method 2, and proceed by computing the integrals. The function is even, so we have to compute only the coefficients a_n. Now,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \cos nx dx = \frac{2}{\pi}\int \frac{1}{2}(1 - \cos 2x)\cos nx dx =$$

$$= \frac{1}{\pi}\int \cos nx dx - \frac{1}{\pi}\int \cos 2x \cos nx dx$$

Let's compute each term separately:

$$\frac{1}{\pi}\int_0^{\pi} \cos nx dx = \begin{cases} \frac{1}{2n}\sin nx\Big|_0^{\pi} = 0, n \neq 0 \\ \frac{1}{\pi}\pi = 1, n = 0 \end{cases}$$

For the second integral use that $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$, so

$$\cos 2x \cos nx = \frac{1}{2}\cos(n+2)x + \frac{1}{2}\cos(n-2)x$$

Now the second integral above is

$$-\frac{1}{\pi}\int \cos 2x \cos nx dx = -\frac{1}{2\pi}\int_0^\pi \cos(n+2)x dx + \frac{1}{2\pi}\int_0^\pi \cos(n-2)x dx =$$

$$= 0 - \frac{1}{2\pi}\int_0^\pi \cos(n-2)x dx =$$

$$= \begin{cases} -\frac{1}{2\pi(n-2)}\sin(n-2)x\Big|_0^\pi = 0, n \neq 2 \\ -\frac{1}{2\pi}\int_0^\pi dx = -\frac{1}{2}, n = 2 \end{cases}$$

Notice – we have to **divide by** n **or** $(n-2)$, which makes sense only for $n \neq 0$,and $n = 2$ and for those you get zero as your integral. For $n = 2$ (and $n = 0$) you compute the integral separately.

5. Consider the polynomials $P_n(x)$ defined by the recursive relation

$$(k+1)P_{k+1} - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, k \geq 1$$

and

$$P_0 = 1,\ P_1(x) = x$$

Write down P_2 and P_3 and show that $P_0,\ P_1,\ P_2$ are orthogonal on the interval $[-1, 1]$.
Skipping some details, the recurrent relation gives you

$$2P_2 - 3xP_1 + P_0 = 0 \Rightarrow P_2 = \frac{1}{2}(3x^2 - 1)$$

and similarly, $P_3 = \frac{1}{2}(5x^3 - 3x)$.
The integration for the orthogonality is routine: you have to check that

$$(P_0, P_1) = \int_{-1}^{1} 1x dx = 0,$$

$$(P_0, P_2) = \int_{-1}^{1} \frac{1}{2}(3x^2 - 1)dx = 0,$$

$$(P_1, P_2) = \int_{-1}^{1} x\frac{1}{2}(3x^2 - 1) = 0$$

Only (P_0, P_2) needs a computation, because the others are integrals of odd functions on a symmetric interval.