Suggested problems for Exam 4

1 Problems from the textbook

For starters, do the core problems and some extras. The extras are in parentheses.

- **19.5**
  5,9,15,18,21,29,32
- **19.6**
  2, (4),(5),6,(8),9,(11),(14),15,(22),(26),31,(32),(33),(34),35, p.887- 36.
- Review your class notes and Ch.7.5, 7.6 from the textbook.
- **12.1**
  (2),(4),5,(6),(9),10,11,12**(Important!)**,13,(14),(15),(16),(17),21.
  Suggestion: For problems 10,11,12 it might be easier to do first the case $p = \pi$ or $p = 1$ to get a feeling why/how does the orthogonality work. Also, I have been calling "L" what the book calls "p".
- **12.2**
  1,5,7,(9),13,17,(18),(19),(20).
  Note: Pay special attention to problems 5 and 17, and after you do them, stare for a while at the result of problem 17, and ask yourself:
  "If I had to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, would I think of taking the Fourier expansion of the function in Problem 5 ???"
  The function
  $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$
  is called *Riemann's zeta function.* Its properties are related to one of the most famous open problems in modern mathematics, the so called *Riemann hypothesis.* Above, you computed that
  $$\zeta(2) = \frac{\pi^2}{6}$$
- **12.3**
  1,(2),(3),5,9,13,(15),25,(27),29,35,43

2 Computing real integrals with the Residue Thm

$$\int_0^\infty \frac{x^2}{x^6+1}dx$$

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$v.p. \int_{-\infty}^\infty \frac{dx}{(x-1)(x^2+4)}$$

$$\int_{-\infty}^\infty \frac{dx}{(x^2+2x+2)^2}$$

$$v.p. \int_{-\infty}^\infty \frac{dx}{x^3+2x^2+2x}$$

$$\int_0^\infty \frac{\cos(ax)}{x^2+1}dx,\ \ a \geq 0$$

$$\int_{-\infty}^\infty \frac{x\ sin\ ax}{x^2+4}dx,\ \ a > 0$$

$$\int_{-\infty}^\infty \frac{\sin ax}{x(x^2+b^2)}dx, a > 0, b > 0$$

3 Orthogonal functions and Fourier series

1. Prove that the rule
$$(z,w) := z\overline{w},\ \ z,w \in \mathbb{C}$$
defines an inner product on the complex vector space $\mathbb{C}$. Prove that
$$(z,w) := Re(z\overline{w})$$
defines an inner product on the real vector space $\mathbb{C}$, aka $\mathbb{R}^2$. Write it out explicitly – have you seen it before?

2. Consider the space
$$P_3(0,1) = \{\text{Polynomials of degree at most 3 with real coefficients on } (0,1)\ \}$$
As you know, it is 4−dimensional and has a basis $\{1, x, x^2, x^3\}$. Construct out of this basis an *orthonormal* basis for $P_3(0,1)$ (orthonormal for the standard inner product).

3. Consider the polynomials:

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Verify that:
a) They are orthogonal on the interval $[-1, 1]$ (with respect to the standard inner product).
b) They satisfy certain differential equations:

$$P_0 \text{ satisfies } (1 - x^2)y'' - 2xy' = 0$$
$$P_1 \text{ satisfies } (1 - x^2)y'' - 2xy' + 2y = 0$$
$$P_2 \text{ satisfies } (1 - x^2)y'' - 2xy' + 6y = 0$$

Do you see a pattern?
c) They satisfy the recurrence relation

$$(k + 1)P_{k+1} - (2k + 1)xP_k(x) + kP_{k-1}(x) = 0$$

d) Using the same recurrence relation as a definition, write down P_4 and P_5. Conditions a),b),c) still hold for P_n, where now b) means that P_n satisfies $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$.
NB The polynomials $P_n(x)$ are called the *Legendre polynomials* .

4. Consider the polynomials $T_n,\ n = 0, 1, 2, \dots$ defined as:

$$T_0(x) = 1, T_1(x) = x,\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n > 1$$

a) Write down the first 4 polynomials.
b) They are orthogonal on $(-1, 1)$, with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. Check the orthogonality of $\{T_0, T_1, T_2\}$.
NB The polynomials $T_n(x)$ are called the *Tschebyshev polynomials.*

5. Consider the polynomials $H_n(x)$, defined in the following way:

$$H_0(x) = 1,\ H_1(x) = x,\ H_n(x) = 2xH_{n-1}(x) - 2(n - 1)H_{n-2}(x), n > 1$$

Write down the first 4 polynomials.
The polynomials H_n are orthogonal on $(-\infty, \infty)$ with respect to the weight function $w(x) = e^{-x^2}$.
NB The polynomials $H_n(x)$ are called the *Hermite polynomials.*
The last three problems are, on one hand, a very good practice for the orthogonality topic, and, on the other, they will be used in our last week of classes. We shall see how to produce orthogonal sets of polynomials from certain boundary problems for ODE next week. You will see many of these polynomials if you take some advanced electrodynamics/quantum mechanics/physical chemistry course, e.g., if you want to solve Schrödinger's equation for the hydrogen atom or for the harmonic oscillator, or to solve the Laplace equation in spherical coordinates, etc.