Suggested problems for Exam 1

1 Starters and appetizers

The core problems:

- **17.1**
  1,5,15,23,27,33,37,39
- **17.2**
  5,9,15,17,23,30,33,39
- **17.3**
  1,5,7,11,15,18,23
- **17.4**
  3,9,17,19,26,29,35,43
- **17.5**
  1,3,9,11,15,17,21,27
- **17.6**
  5,11,15,21,25,31,36,41,43,47 (Ignore the problems about Logarithms)

2 More problems from the textbook

- **17.1**
  13,14,16,30,32,35,36,40
- **17.2**
  3,11,20,21,33
- **17.3**
  9,13,24,25,26
- **17.4**
  10,13,14,19,24,25,34, 38,44
- **17.5**
  2,4,6,13,18, 19,29

3 Algebraic Operations with Complex Numbers

1) Write the numbers

$$1, -1, i, -i, 1+i, 1-i, -1+i, \frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}, \frac{2}{1-i\sqrt{3}}, \frac{1-i}{1+i}$$

in trigonometric form,i.e., in the form $re^{i\phi}$.
2) How many distinct numbers are there in the sequence:

$$a_n = \left(\frac{1+i\sqrt{3}}{2}\right)^n, n = 0, \pm 1, \pm 2, \dots$$

3) Find the roots:

$$\sqrt{i},\ \sqrt[3]{-i},\ \sqrt[8]{-1},\ \sqrt[5]{1-i},\ \sqrt[7]{\frac{1}{1-i}},\ \sqrt[3]{\frac{1+i\sqrt{3}}{2}},\ \sqrt[4]{\frac{2}{1-i\sqrt{3}}}$$

using de Moivre's formula.
4) Let the function $e^{i\alpha}$ be defined as: $e^{i\alpha} := \cos\alpha + i\ \sin\alpha$ for all real α. Prove *Euler's formulas*:

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}, \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

5) Prove that $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$.
6) Prove that the complex conjugate of a sum, difference, product, and ratio of two complex numbers is the sum, difference, product, and ratio of the complex conjugates of these numbers.
7) Is the equality

$$Re(zw) = Re(z)Re(w)$$

(always) true?
8)Write the numbers $\frac{1}{i}, \frac{1}{1-i}$ in the form $a + ib, a, b \in \mathbb{R}$.

4 Some geometry with complex numbers

1) Give three different descriptions of the unit circle in $\mathbb{C}$.
2) Draw all the solutions of the equations $z^3 - 1 = 0$,$z^5 - 1 = 0$,$z^n - 1 = 0$.
3) Let $a, b \in \mathbb{C}$. Describe the set of complex numbers

$$z = \lambda a + (1-\lambda)b,\ \ \lambda \in [0, 1]$$

Is it closed, open, neither, both? Draw a picture.
4) Describe the following subsets of the complex plane, and for each of them determine if it is open/closed/both/neither and draw a picture.

$$0 \le Imz \le 1$$

$$Re z < 3$$
$$Re \frac{1}{z} > 0$$
$$Re \frac{1}{z} = 1$$
$$|z - a| = R, R > 0$$
$$|z - a| \geq R, R \geq 0$$
$$|z - i| + |z + i| = R, R > 0$$
$$|z - i| - |z + i| = R, R > 0$$

5) Show that every circle is given by an equation of the form

$$z\bar{z} + lz + \bar{l}\bar{z} + m = 0$$

where m is a real number and $m < |l|^2, l \in \mathbb{C}$. Conversely, show that every such equation is an equation of a circle.
6) Show that

$$z = a + Re^{i\phi},\ 0 \leq \phi \leq 2\pi$$

is a parametric equation of a circle with centre $a \in \mathbb{C}$ and radius R.
7) Prove that the equation

$$Az - \bar{A}\bar{z} + B = 0, A \neq 0, ReB = 0$$

is a cartesian equation of line.

5 Differentiability and Analyticity

1) *Without* using the Cauchy-Riemann equations prove that the function $f(z) = Rez$ is not differentiable at any point, while the function $g(z) = |z|^2$ has $\mathbb{C}$-derivative only at $z = 0$.
2) When do we call a function *analytic* ? Give a sufficient condition for analyticity. When is an $\mathbb{R}$-differentiable function $\mathbb{C}$-differentiable? When is a $\mathbb{C}$-differentiable function $\mathbb{R}$-differentiable?
3) Find the points of $\mathbb{C}$-differentiability and analyticity of the function

$$f(z) = \overline{z}(z - 1)(z - 2)$$

4) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = \langle u(x, y), v(x, y)\rangle$, where $u = x^3 + y\sin(x), v = x\tan(y)$. Compute the differential *df* at the points $(0, 0)$ and $(1, 1)$. Find the linearization of f at these points, i.e., a linear function $\mathbb{R}^2 \to \mathbb{R}^2$, which approximates f up to first order. Where is it continuous? $\mathbb{R}$-differentiable? $\mathbb{C}$-differentiable? Analytic?
5) Prove that if *both* $f(z)$ and $\overline{f(z)}$ are analytic in a domain,G, then f is a constant.
6) Given the function $u(x, y)$, find an analytic function $f = u + iv$, i.e., find a

harmonic conjugate of u, where:

$$u(x, y) = 2x(1 - y)$$
$$u(x, y) = \sinh(x)\sin(y)$$