Sec 10.7 p600 #6

$\sum_{n=0}^{\infty} (2x)^n$ geometric with $r = 2x$, so series converges (absolutely) for $-1 < 2x < 1$, ie $-\frac{1}{2} < x < \frac{1}{2}$ and diverges otherwise

(interval of conv: $(-\frac{1}{2}, \frac{1}{2})$, radius $= \frac{1}{2}$)

Sec 10.7 p600 #10

$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(x-1)^n} \right| = \lim_{n\to\infty} |x-1| \frac{\sqrt{n}}{\sqrt{n+1}}$$

$= |x-1| < 1$ means $0 < x < 2$

At $x=0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally (A.S. test, p-series $p = 1/2$)

At $x=2$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

So interval of convergence is $[0, 2)$
radius of convergence is 1.

Sec 10.7 p600 #14

$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$: RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)^3 3^{n+1}} \cdot \frac{n^3 3^n}{(x-1)^n} \right| = \lim_{n\to\infty} \frac{|x-1|}{3} \frac{n^3}{(n+1)^3}$$

$= \frac{|x-1|}{3} < 1$ means $-2 < x < 4$

At $x = -2$: $\sum \frac{(-3)^n}{n^3 3^n} = \sum \frac{(-1)^n}{n^3}$
converges absolutely by A.S. test, p-series $p=3$

At $x = 4$: $\sum \frac{3^n}{n^3 3^n} = \sum \frac{1}{n^3}$ converges (abs.)

So interval of convergence is $[-2, 4]$.
Radius of convergence is 3

Sec 10.7 p600 #28

$\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$ RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n (n+1)(x-1)^n} \right| = \lim_{n\to\infty} 2|x-1| \frac{n+2}{n+1}$$

$= 2|x-1| < 1$ means $\frac{1}{2} < x < \frac{3}{2}$

At $x = \frac{1}{2}$: $\sum_{n=0}^{\infty} (n+1)$ diverges (nth term test)

At $x = \frac{3}{2}$: $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$ diverges

So interval of convergence is $(\frac{1}{2}, \frac{3}{2})$
Radius of convergence is $1/2$

Sec 10.7 p600 #34

$\sum_{n=1}^{\infty} \frac{3\cdot5\cdot7\cdots(2n+1)}{n^2 2^n} x^{n+1}$ RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)\, x^{n+2}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 \cdot 2^n}{3\cdot5\cdot7\cdots(2n+1)\, x^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+3)|x|}{2} \frac{n^2}{(n+1)^2} = \infty$$

So the series converges only for $x = 0$
Radius of convergence is 0.

Sec 10.7 p600 #38

$\sum_{n=1}^{\infty} \left(\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)} \right)^2 x^n$ RATIO TEST

$$\lim_{n\to\infty} \left| \left(\frac{2\cdot4\cdot6\cdots(2n)(2n+2)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)} \right)^2 \frac{x^{n+1}}{x^n} \left(\frac{2\cdot5\cdot8\cdots(3n-1)}{2\cdot4\cdot6\cdots(2n)} \right)^2 \right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)^2}{(3n+2)^2} |x| = \frac{4}{9}|x|$$

so series converges (at least) for $-\frac{9}{4} < x < \frac{9}{4}$ and radius of convergence is $\frac{9}{4}$.

Sec 10.7 p600 #46

$\sum_{n=0}^{\infty} (\ln x)^n$: geometric series, $a = 1$, $r = \ln x$

converges to $\frac{1}{1 - \ln x}$ provided $-1 < \ln x < 1$,

ie. for $\frac{1}{e} < x < e$.

Sec 10.7 p600 #52

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

a) $$\frac{d}{dx}e^x = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= e^x \;!!$$

b) $$\int e^x dx = \int 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots dx$$
$$= C + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= (C-1) + e^x.$$

c) $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$e^x \cdot e^{-x} = 1 + (1-1)x + \left(\tfrac{1}{2} - 1 + \tfrac{1}{2}\right)x^2$$
$$+ \left(\tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6}\right)x^3$$
$$+ \left(\tfrac{1}{24} - \tfrac{1}{6} + \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{24}\right)x^4$$
$$+ \left(\tfrac{1}{120} - \tfrac{1}{24} + \tfrac{1}{12} - \tfrac{1}{12} + \tfrac{1}{24} - \tfrac{1}{120}\right)x^5$$
$$+ \cdots$$
$$= 1 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 + \cdots$$

Sec 10.8 p606 #8

$f(x) = \tan x \quad a = \pi/4$ $\qquad f(\tfrac{\pi}{4}) = 1$

$f'(x) = \sec^2(x)$ $\qquad f'(\tfrac{\pi}{4}) = 2$

$f''(x) = 2\sec^2(x)\tan x$ $\qquad f''(\tfrac{\pi}{4}) = 4$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$ $\qquad f'''(\tfrac{\pi}{4}) = 16$

degree 0: $P_0 = 1$

degree 1: $P_1 = 1 + 2(x - \tfrac{\pi}{4})$

degree 2: $P_2 = 1 + 2(x-\tfrac{\pi}{4}) + 2(x-\tfrac{\pi}{4})^2$

degree 3: $P_3 = 1 + 2(x-\tfrac{\pi}{4}) + 2(x-\tfrac{\pi}{4})^2 + \tfrac{8}{3}(x-\tfrac{\pi}{4})^3$

Sec 10.8 p606 #12

$f(x) = xe^x \; : \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

So $$xe^x = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

Sec 10.8 p608 #22

$$f(x) = \frac{x^2}{x+1}$$

Recall $\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \cdots$ Let $\square = -x$,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$
$$\frac{x^2}{1+x} = x^2(1 - x + x^2 - x^3 + \cdots)$$
$$= x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=2}^{\infty} (-1)^n x^n.$$

Sec 10.8 p608 #24

$f(x) = 2x^3 + x^2 + 3x - 8 \qquad a = 1 \quad f(1) = -2$

$f'(x) = 6x^2 + 2x + 3 \qquad f'(1) = 11$

$f''(x) = 12x + 2 \qquad f''(1) = 14$

$f'''(x) = 12 \qquad f'''(1) = 12$

$f^{(n)}(x) = 0$ for $n \geq 4 \qquad f^{(n)}(1) = 0, n \geq 4$

So $f(x) = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3.$

Sec 10.8 p608 #32

$f(x) = \sqrt{x+1} = (x+1)^{1/2}, \; a = 0$

$f(x) = (x+1)^{1/2}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(x+1)^{-1/2}$ $\qquad f'(0) = 1/2$

$f''(x) = \frac{-1}{2^2}(x+1)^{-3/2}$ $\qquad f''(0) = -1/2^2$

$f'''(x) = +\frac{1 \cdot 3}{2^3}(x+1)^{-5/2}$ $\qquad f'''(0) = \frac{1 \cdot 3}{2^3}$

$f''''(x) = \frac{-1 \cdot 3 \cdot 5}{2^4}(x+1)^{-7/2}$ $\qquad f''''(0) = \frac{-1 \cdot 3 \cdot 5}{2^4}$

$\vdots$ $\qquad \vdots$

$$f^{(n)}(x) = \frac{(-1)^{n+1} 3 \cdot 5 \cdots (2n-3)}{2^n}(x-1)^{-\frac{(2n-1)}{2}}$$

$$f^{(n)}(0) = \frac{(-1)^{n+1} 3 \cdot 5 \cdots (2n-3)}{2^n}$$

($n = 0, 1$ do not fit this pattern)

$$\Rightarrow \sqrt{x+1} = 1 + \tfrac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdots (2n-3)}{2^n \, n!} x^n$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots$$

Sec 10.9 p 613 # 8

arctan$(3x^4)$.

we know

$$\arctan\square = \square - \frac{\square^3}{3} + \frac{\square^5}{5} - \cdots$$

So

$$\arctan(3x^4) = 3x^4 - \frac{(3x^4)^3}{3} + \frac{(3x^4)^5}{5} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{8n+4}}{2n+1}$$

Sec 10.9 p613 # 16

$x^2 \cos x^2$

$$\cos\square = 1 - \frac{\square^2}{2!} + \frac{\square^4}{4!} - \cdots$$

so

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$$

and

$$x^2\cos x^2 = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!}$$

Sec 10.9 p613 # 20

$x\ln(1+2x)$

$$\ln(1+\square) = \square - \frac{\square^2}{2} + \frac{\square^3}{3} - \frac{\square^4}{4} + \cdots$$

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots$$

$$x\ln(1+2x) = 2x^2 - \frac{2^2x^3}{2} + \frac{2^3x^4}{3} - \frac{2^4x^5}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n x^{n+1}}{n}$$

Sec 10.9 p 613 # 30

$$\frac{\ln(1+x)}{1-x} = (1+x+x^2+x^3+x^4+\cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= x + \left(1-\tfrac{1}{2}\right)x^2 + \left(1 - \tfrac{1}{2} + \tfrac{1}{3}\right)x^3 + \left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4}\right)x^4 + \cdots$$

$$= x + \tfrac{1}{2}x^2 + \tfrac{5}{6}x^3 + \tfrac{7}{12}x^4 + \cdots$$

Sec 10.9 p613 # 36

$R_4 = \dfrac{M(\frac{1}{2})^5}{5!}$ where M is the max of e^x on $[0, \frac{1}{2}]$

We know $e < 3$ so $e^x < e^{1/2} < 3^{1/2} < 1.75$ on $[0, \frac{1}{2}]$

An estimate of the error is thus:

$$R_4 < \frac{1.75}{2^5 \cdot 5!} = \frac{7}{4 \cdot 32 \cdot 120} = \frac{7}{128 \cdot 120} \approx 0.00046$$

Sec 10.9 p613 # 40

Error estimate is $R_1 < \dfrac{M(0.01)^2}{2!}$ where M is the max of $\left|\dfrac{d^2}{dx^2}\sqrt{1+x}\right|$ on $[-0.01, 0.01]$

$$\frac{d}{dx}(1+x)^{1/2} = \tfrac{1}{2}(1+x)^{-1/2}, \quad \frac{d^2}{dx^2}(1+x)^{1/2} = -\tfrac{1}{4}(1+x)^{-3/2}$$

Max of this occurs when $x = -0.01$: $\frac{1}{4}(0.99)^{-3/2}$

So $R_1 < \dfrac{\frac{1}{4}(0.99)^{-3/2}(0.01)^2}{2}$

Now $(0.99)^{-3/2} < (0.81)^{-3/2} = (0.9)^{-3} = \dfrac{1}{0.729} < 1.5$

So $R_1 < \dfrac{1.5\,(0.01)^2}{8}$

$= 0.00001875$

Sec 10.9 p613 # 50

(a) First, note the Taylor expansion of $\sin x$ around $a = \pi$ is

$$\sin x = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \cdots$$

So if P is near π we have

$\sin P = -(P-\pi) + \cdots$ and error estimate (alternating series) is

$$\left|\sin P - (-(P-\pi))\right| < \left|\frac{(P-\pi)^3}{3!}\right|$$

ie $\left|(\sin P + P) - \pi\right| < \left|\dfrac{(P-\pi)^3}{6}\right|$

If $|P-\pi| < 10^{-n}$, then

$$|P + \sin P - \pi| < \tfrac{1}{6}\,10^{-3n}$$

ie 3 times as many correct decimals

b) Try with $n=2$:

$3.14 + \sin(3.14) =$

$3.14 + 0.001592653\cdots$

$= 3.1415926533$

$\pi = 3.1415926534\cdots$!!

Sec 10.10 p620 #8

$(1+x^2)^{-1/3}$

Binomial

$$(1+\square)^{-1/3} = 1 - \tfrac{1}{3}\square + \frac{\frac{1}{3}\cdot\frac{4}{3}}{2!}\square^2 - \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{3!}\square^3 + \cdots$$

$$(1+x^2)^{-1/3} = 1 - \tfrac{1}{3}x^2 + \frac{1\cdot 4}{3^2\cdot 2!}x^4 - \frac{1\cdot 4\cdot 7}{3^3\cdot 3!}x^6 + \cdots$$

Sec 10.10 p620 #16

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx = \int_0^{0.2} \frac{1}{x}\left(-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)dx$$

$$= \int_0^{0.2}\left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \cdots\right)dx =$$

$$= -x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!} - \cdots \Big|_0^{0.2}$$

Now $\frac{0.2^3}{3\cdot 3!} = \frac{0.008}{18} < 0.001$

So to nearest 0.001, integral is

$$-0.2 + \frac{0.2^2}{2\cdot 2!} = -0.2 + \frac{0.04}{4} = -0.190$$

Sec 10.10 p620 #20

$$\int_0^{0.1} e^{-x^2}dx = \int_0^{0.1} 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \cdots \Big|_0^{0.1}$$

Now $\frac{(0.1)^7}{7\cdot 3!} = \frac{10^{-7}}{42} < 10^{-8}$

so to within 10^{-8}, integral is

$$0.1 - \frac{0.1^3}{3} + \frac{0.1^5}{10} = \begin{array}{r} 0.100010 \\ -0.000333 \\ \hline 0.099677 \end{array}$$

Sec 10.10 p620 #24

The next term in the approximation is

$\int_0^1 \frac{t^4}{8!}dt = \frac{1}{5\cdot 8!}$ The error is less than this

Sec 10.10 p620 #34

$$\lim_{y\to 0} \frac{\arctan y - \sin y}{y^3\cos y}$$

$$= \lim_{y\to 0} \frac{\left(\cancel{y} - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right) - \left(\cancel{y} - \frac{y^3}{6} + \frac{y^5}{120} - \cdots\right)}{y^3\left(1 - \frac{y^2}{2} + \cdots\right)}$$

$$= \lim_{y\to 0} \frac{y^3\left(-\frac{1}{6} + \cdots\right)}{y^3(1 - \cdots)} = -\frac{1}{6}.$$

Sec 10.10 p620 #42

$$\left(\tfrac{1}{4}\right)^3 + \left(\tfrac{1}{4}\right)^4 + \left(\tfrac{1}{4}\right)^5 + \left(\tfrac{1}{4}\right)^6 + \cdots$$

geometric – 1st term is $\frac{1}{64}$, $r = 1/4$

$$\text{Sum} = \frac{1/64}{1-1/4} = \frac{1/64}{3/4} = \frac{1}{48}$$

Sec 10.10 p620 #46

$$\frac{2}{3} - \frac{2^3}{3^3\cdot 3} + \frac{2^5}{3^5\cdot 5} - \frac{2^7}{3^7\cdot 7}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2/3)^{2n+1}}{2n+1}$$

$$= \arctan 2/3$$

Sec 10.10 p620 #58

By long division, $\tan t = \frac{\sin t}{\cos t}$:

$$\begin{array}{r|l} & t + t^3/3 + \frac{2t^5}{15} \\ \hline 1 - \frac{t^2}{2} + \frac{t^4}{24} - \cdots & t - \frac{t^3}{6} + \frac{t^5}{120} - \cdots \\ & t - \frac{t^3}{2} + \frac{t^5}{24} \\ \hline & \frac{t^3}{3} - \frac{t^5}{30} \\ & \frac{t^3}{3} - \frac{t^5}{6} \\ \hline & 2t^5/15 \end{array}$$

So $\tan t = t + \frac{t^3}{3} + \frac{2t^5}{15} + \cdots$

So $\ln(\sec x) = \int_0^x t + \frac{t^3}{3} + \frac{2t^5}{15} + \cdots dt$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$$