MATH 104 – Practice Problems for Exam 2

1. Find the area between:

   (a) $x = 0$, $y = 1/\sqrt{1+x^2}$, $y = x/\sqrt{2}$

   Answer: $\ln(1+\sqrt{2}) - \dfrac{\sqrt{2}}{4}$

   (b) $y = 3e^{2x}$, $y = xe^{2x}$, $x = 0$

   Answer: $\dfrac{e^6}{4} - \dfrac{7}{4}$

   (c) $y = \dfrac{x}{x^2-1}$ and the x axis, for $2 \le x \le 4$.

   Answer: $\dfrac{\ln 5}{2}$

2. Calculate the volume obtained by rotating:

   (a) The region in problem 1a around the x-axis

   Answer: $\dfrac{\pi^2}{4} - \dfrac{\pi}{6}$

   (b) The region in problem 1a around the y-axis

   Answer: $2\pi\left(\dfrac{5\sqrt{2}}{6} - 1\right)$

   (c) The region in problem 1b around the x-axis

   Answer: $\pi\left(\dfrac{11e^{12}}{32} - \dfrac{71}{32}\right)$

   (d) The region in problem 1b around the y-axis
   Answer: $\pi(e^6+2)$

   (e) The region in problem 1c around the x-axis
   Answer: $\pi(\frac{1}{2}\ln 3 - \frac{1}{4}\ln 5 + \frac{1}{5})$

   (f) The region in problem 1c around the y-axis
   Answer: $\pi(2\ln 3 - \ln 5 + 4)$

   (g) The region in problem 1c around the line $x = 1$
   Answer: $2\pi(\ln 3 - \ln 5 + 2)$

   (h) The region in problem 1c around the line $y = -1$
   Answer: $\pi(\frac{1}{2}\ln 3 + \frac{3}{4}\ln 5 + \frac{1}{5})$

3. Integrate: (straightforward)

   (a) $\displaystyle\int x^4 e^{2x}\,dx$

   Answer: $\frac{1}{4}e^{2x}(2x^4 - 4x^3 + 6x^2 - 6x + 3) + C$

(b) $\int x^2 \tan^{-1}(x)\,dx$

Answer: $\frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$

(c) $\int \frac{x}{x^2 - 5x + 4}\,dx$

Answer: $\frac{4}{3}\ln(x-4) - \frac{1}{3}\ln(x-1) + C$

(d) $\int \sqrt{1+4x^2}\,dx$

Answer: $\frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln(2x + \sqrt{1+4x^2}) + C$

(e) $\int \frac{1}{1+\sqrt{x}}\,dx$

Answer: $2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$

(f) $\int \frac{\cos^2\sqrt{x}}{\sqrt{x}}\,dx$

Answer: $\sqrt{x} + \frac{1}{2}\sin(2\sqrt{x}) + C$

(g) $\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$

Answer:$\sec(\ln x) + C$

4. Integrate: (trickier)

(a) $\int \sin^4(2x)\,dx$

Answer: $\frac{3}{8}x - \frac{3}{16}\cos(2x)\sin(3x) - \frac{1}{8}\cos(2x)\sin^3(2x) + C$

(b) $\int \frac{\sqrt{x^2-25}}{x}\,dx$

Answer: $\sqrt{x^2-25} + 5\arcsin(5/x^2) + C$

(c) $\int \frac{e^t}{e^{2t}-4}\,dt$

Answer: $\frac{1}{4}(\ln(e^t - 2) - \ln(e^t + 2)) + C$

(d) $\int \sqrt{1+e^x}\,dx$

Answer: $2\sqrt{1+e^x} + \ln(\sqrt{1+e^x} - 1) - \ln(\sqrt{1+e^x} + 1) + C$

(e) $\int e^{\sqrt{x}}\,dx$

Answer: $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

5. Evaluate:

(a) $\int_e^\infty \frac{1}{x(\ln x)^3}\,dx$

Answer: $1/2$

(b) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+4)}$
Answer: $\pi/2$

(c) $\displaystyle\int_0^1 \sqrt{\frac{1-y}{y}}\,dy$
Answer: $\pi/2$

6. Find the general solution to each of the following differential equations:

   (a) $x\,\dfrac{dy}{dx} = y^2$
   Answer: $y = 1/(C - \ln(x))$

   (b) $(x^2+1)\dfrac{dy}{dx} = y$
   Answer: $y = Ce^{\arctan x}$

7. Find the specific solution of each equation that satisfies the given condition:

   (a) $\dfrac{dy}{dx} = xy, \qquad y(1) = 3$
   Answer: $y = 3e^{(x^2-1)/2}$

   (b) $\dfrac{dy}{dx} = xy + x, \qquad y(0) = 10$
   Answer $y = 11e^{x^2/2} - 1$

8. In a second-order chemical reaction, the reactant A is used up in such a way that the amount of it present decreases at a rate proportional to the square of the amount present. Suppose this reaction begins with 50 grams of A present, and after 10 seconds there are only 25 grams left. How long after the beginning of the reaction will there be only 10 grams left? Will all of the A disappear in a finite time, or will there always be a little bit present?
   Answer: 40 seconds, and there will always be a little bit present.

9. According to Newton's law of heating and cooling, if the temperature of an object is different from the temperature of its environment, then the temperature of the object will change so that the difference between the object's temperature and the ambient temperature decreases at a rate proportional to this difference.

   On a hot day, a thermometer was brought outdoors from an air-conditioned building. The temperature inside the building was 21° C, and so this is what the thermometer read at the moment it was brought outside. One minute later the thermometer read 27° C, and a minute after that it read 31° C. What was the temperature outside? (Impress us and express the answer without using logarithms or the number e.)
   Answer: 39° C

10. A super-fast-growing bacteria reproduces so quickly that the rate of production of new bacteria is proportional to the *square* of the number already present. If a sample starts with 100 bacteria, and after 3 hours there are 200 bacteria, how long (after the starting time) will it take until there are (theoretically) an *infinite* number of bacteria?
Answer: 6 hours

11. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx =$

(a) $\frac{2\pi}{15}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{5}$ (d) $\frac{\pi}{2}$ (e) $\frac{3\pi}{5}$ (f) 2π
Answer: B

12. $\displaystyle\int_0^\infty x^2 e^{-2x}\,dx =$

(a) 1/4 (b) 4/3 (c) 2 (d) 3/8 (e) $e^{1/2}$ (f) diverges
Answer: A

13. $\displaystyle\int_4^6 \frac{x^3-6x-4}{x^2-x-6}\,dx =$

(a) $2+\ln(3)$ (b) $4/3$ (c) $12+\ln(3)$
(d) $47/24$ (e) $2+\ln(4/3)$ (f) $10+\ln(3)$
Answer: C

14. What is the volume of the solid obtained by rotating the region between the graph of $y = \dfrac{1}{x^2+4x+3}$ and the x-axis for $0 \le x \le 1$ around the y-axis?

(a) $\pi(\ln 2 + 2\ln 3)$ (b) $2\pi(4\ln 3 - 5\ln 2)$ (c) $2\pi\ln 12$
(d) $\pi(2\ln 3 + 3\ln 2)$ (e) $2\pi\ln 18$ (f) $\pi(5\ln 2 - 3\ln 3)$
Answer: F

15. $\displaystyle\int_0^\infty \frac{1}{x^2+2x+2}\,dx =$

(a) 1 (b) π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$ (e) $\frac{\pi}{2}-1$ (f) diverges
Answer: D

16. Find the surface area of the surface obtained by revolving the part of the graph of $y = x^3/9$ where $0 \le x \le 2$ around the x-axis.

(a) $\frac{38\pi}{27}$ (b) $\frac{121\pi}{72}$ (c) $\frac{76\pi}{9}$ (d) $\frac{77\pi}{48}$ (e) $\frac{98\pi}{81}$ (f) $\frac{86\pi}{27}$
Answer: E

17. Solve the initial-value problem: $\dfrac{dy}{dx} = e^y \sin x,\ y(0) = 0.$

(a) $y = \ln(\sec x)$ (b) $\dfrac{1}{2}\ln(\cos x)$ (c) $\dfrac{\pi}{4} + \ln(\cos x)$

(d) $\dfrac{\pi}{4}\ln(\sec x)$ (e) $\dfrac{1}{2}\ln(\sec x)$ (f) $\ln(\cos x)$

Answer: A

18. The function

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for a certain value of k. Find the *mean* of that probability density function.

(a) $\dfrac{3}{4}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{4}$ (d) $\dfrac{2}{3}$ (e) $\dfrac{1}{3}$ (f) 2

Answer: D

19. If water leaks out of a small hole in a cylindrical bucket, then the height of the water level above the bottom of the bucket decreases at a rate proportional to the *square root* of the height. If the water level starts out at a height of 25 cm, and if after 10 minutes it is down to 16 cm, how long after the start will the bucket be empty?

(a) 30 min (b) 40 min (c) 50 min

(d) 60 min (e) 70 min (f) the bucket will never be completely empty

Answer: C

1. What is the length of the part of the curve $y = x^{1/2} - \dfrac{x^{3/2}}{3}$ between $x = 0$ and $x = 2$?

(a) $\dfrac{5}{3}\sqrt{2}$ (b) $\dfrac{7}{3}\sqrt{2}$ (c) $\dfrac{7}{6}\sqrt{2}$ (d) $\dfrac{25}{6}\sqrt{2}$ (e) $\dfrac{11}{3}\sqrt{2}$ (f) $\dfrac{11}{6}\sqrt{2}$

Answer: A

2. $\displaystyle\int_0^e x^4 \ln x \, dx =$

(a) $\dfrac{3e^4}{16}$ (b) $\dfrac{4e^5}{25}$ (c) $\dfrac{5e^6}{36}$ (d) $\dfrac{6e^7}{49}$ (e) $\dfrac{7e^8}{64}$ (f) $\dfrac{8e^9}{81}$

Answer: B

3. $\displaystyle\int_1^\infty \frac{\sqrt{1+x^2}}{x^6}\, dx =$

(a) $\dfrac{2}{15}(\sqrt{2}+1)$ (b) $\dfrac{1}{15}(4\sqrt{2}+1)$ (c) $\dfrac{1}{3}(2\sqrt{2}-1)$

(d) $\frac{2}{5}(\sqrt{2}-1)$ (e) $\frac{1}{5}(4\sqrt{2}-1)$ (f) diverges

Answer: A

4. $\int_3^4 \frac{x}{x^2-6x+5}\,dx =$

(a) $\frac{1}{2}\ln 3 - 3\ln 2$ (b) $\frac{1}{4}\ln 3 - \ln 2$ (c) $-\frac{1}{4}\ln 3 - 3\ln 2$

(d) $-\frac{1}{6}\ln 3 - \ln 2$ (e) $-\frac{1}{4}\ln 3 - \ln 2$ (f) $-\frac{1}{2}\ln 3 - \ln 2$

Answer: E

5. What is the surface area of the surface obtained by rotating the part of the curve $y = \frac{2}{3}x^3$ for $0 \le x \le 1$ around the x-axis?

(a) $\frac{\pi}{18}(5\sqrt{5}-1)$ (b) $\frac{\pi}{21}(7\sqrt{7}-1)$ (c) $\frac{\pi}{24}(8\sqrt{8}-1)$

(d) $\frac{\pi}{27}(10\sqrt{10}-1)$ (e) $\frac{\pi}{36}(17\sqrt{17}-1)$ (f) $\frac{\pi}{45}(21\sqrt{21}-1)$

Answer: A

6. Let $y(x)$ be the solution of initial-value problem $y' + 4xy = 0$, $y(0) = 1$. Then $y(2) =$

(a) e^{-2} (b) e^{-4} (c) e^{-6} (d) e^{-8} (e) e^{-10} (f) e^{-12}

Answer: D

7. Some enterprising Penn scientists have created a sample of Unobtanium in their lab. One of the remarkable properties of this material is that when it is heated, contrary to Newton's law of cooling, its temperature decreases to room temperature at a rate proportional to the *square root* of the difference between its temperature and the ambient temperature. In a laboratory kept at 20 degrees C, the sample is heated to a temperature of 36 degrees C. After 2 minutes have passed, the temperature of the sample is 29 degrees C. How long after the initial heating will the sample's temperature be equal to the room temperature?

(a) 6 minutes (b) 8 minutes (c) 10 minutes

(d) 12 minutes (e) 14 minutes (f) 16 minutes

Answer B

8. $\int_0^{\pi/8} \tan 4t\,dt =$

(a) 1 (b) $\ln 2$ (c) $\frac{1}{2}\ln 2$ (d) $1-\frac{1}{2}\ln 2$ (e) $\sqrt{2}-\ln 2$ (f) diverges

Answer: F

9. The function

$$f(x) = \begin{cases} kxe^{-4x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for a certain value of k. Find the *mean* of that probability density function.

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{5}$ (e) $\frac{1}{3}$ (f) $\frac{1}{8}$

Answer: C

10. The functions $y_1(t)$ and $y_2(t)$ are both solutions of the autonomous differential equation $\frac{dy}{dt} = 3\sin\left(\frac{y}{2}\right)$ but satisfy different initial conditions: $y_1(0) = 1$ and $y_2(0) = -1$. Either by solving the differential equation or, better, by thinking about its geometry (slope field), calculate

$$\lim_{t\to\infty} (y_1(t) - y_2(t)).$$

(a) 0 (b) 2π (c) 4π (d) 6π (e) 8π (f) ∞

Answer: C