

MATH 104 – Practice Problems for Final Exam - Selected (so far) Hints and answers

1(a) The two curves intersect at $(0,0)$ and at $(\frac{1}{2}, 3)$. And $3 \sin(\pi x) > 6x$ for $0 < x < 1/2$, so the area is given by the (elementary) integral

$$\int_0^{1/2} 3 \sin(\pi x) - 6x \, dx = \frac{3}{4} \frac{4 - \pi}{\pi}.$$

1(b) This is a trig substitution integral, let $x = 2 \tan \theta$. You'll need a triangle. The result is $\ln((3 + \sqrt{13})/2)$.

1(c) This is easy – integrate $\sqrt{x} - x$ from 0 to 1.

2(a) To do this one, compute $\pi \int_0^{1/2} (3 \sin(\pi x))^2 - (6x)^2 \, dx$, which is $3\pi/4$.

2(b) To do this one, compute $2\pi \int_0^{1/2} x(3 \sin(\pi x) - 6x) \, dx$, which is $6/\pi - \pi/2$.

2(c) To do this one, compute $\pi \int_0^3 \frac{1}{x^2 + 4} \, dx$, which is $\frac{\pi}{2} \arctan(\frac{3}{2})$.

2(d) To do this one, compute $2\pi \int_0^3 \frac{x}{\sqrt{x^2 + 4}} \, dx$, which is $2\pi(\sqrt{13} - 2)$.

2(e) Washers: integrate $\pi(x - x^4)$ from 0 to 1.

2(f) Shells: integrate $2\pi x(\sqrt{x} - x^2)$ from 0 to 1. Why is the answer the same as the previous problem?

2(g) Shells: integrate $2\pi(1 - x)(\sqrt{x} - x^2)$ from 0 to 1.

2(h) Washers: integrate $\pi((\sqrt{x} + 1)^2 - (x^2 + 1)^2)$ from 0 to 1.

3(a) Answer: $(\frac{\sqrt{5}}{2} + \frac{1}{8} \ln(9 + 4\sqrt{5}))$.

3(b) Answer: $\frac{58\sqrt{58}}{27} - \frac{13\sqrt{13}}{27}$.

3(c) Answer: infinite (the graph goes to negative infinity at $x = 0$).

3(d) Answer: $\ln(2 + \sqrt{3}) - \frac{1}{2} \ln(2) + \ln(2 - \sqrt{2})$

3(e) Answer: $\frac{1}{2} \ln(2 + \sqrt{2}) - \frac{1}{2} \ln(2 - \sqrt{2}) + \sqrt{1 + e^2} - \sqrt{2} + \frac{1}{2} \ln(\sqrt{1 + e^2} - 1) - \frac{1}{2} \ln(\sqrt{1 + e^2} + 1)$.

4(a) Answer: $\frac{\pi}{6}(5\sqrt{5} - 1)$.

4(b) Answer: $\frac{\pi}{64}(36\sqrt{5} - \ln(9 + 4\sqrt{5}))$.

4(e) Answer: $\pi(e\sqrt{1+e^2} - \sqrt{2} + \ln(e + \sqrt{1+e^2}) - \ln(1 + \sqrt{2}))$.

5(a) This is integration by parts, let $u = \ln(2x)$ and $dv = x^4 dx$. The answer is $\frac{1}{5}x^5 \ln(2x) - \frac{1}{25}x^5 + C$.

5(b) Parts again, this time $u = x^2$ and $dv = \cos(3x) dx$. You'll have to do parts twice, or else use tabular integration. The result is $\frac{1}{3}x^2 \sin(3x) + \frac{2}{9}x \cos(3x) - \frac{2}{27} \sin(3x) + C$.

5(c) Partial fractions – the denominator factors as $(x + 3)(x + 5)$ and the result is $\frac{3}{2} \ln(x + 5) - \frac{1}{2} \ln(x + 3) + C$.

5(d) Trig substitution – since $4x^2$ should equal $\sin^2 \theta$, let $x = \frac{1}{2} \sin \theta$. The resulting integral has $\cos^2 \theta$, and so you'll have to use the trig identity for that. The answer is $\frac{1}{2}x\sqrt{1-4x^2} + \frac{1}{4} \arcsin(2x) + C$. (You could probably also do this by parts, letting $u = \sqrt{1-4x^2}$ and $dv = dx$.)

5(e) Start with a desparation substitution $u = \sqrt{x}$, or $x = u^2$ with $dx = 2u du$, so the integral becomes

$$\int \frac{2u}{u-1} du.$$

Divide it out, then it's easy. The result is $2\sqrt{x} + 2 \ln(\sqrt{x} - 1) + C$.

5(f) First let $u = \ln x$, then you have to integrate $\sin^2 u$ using the trig identity. The result is $\frac{1}{2} \ln x - \frac{1}{4} \sin(2 \ln x) + C$.

5(g) Lots of substitutions – $u = \ln x$ changes it to the integral of $\sec^u / \sqrt{1 - \tan u}$. Then let $v = \tan u$ to get the integral of $1/\sqrt{1-u}$. Then it's not too bad. The result is $-2\sqrt{1 - \tan(\ln x)} + C$.

6(a) Use the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$, multiply out, and use the identity again.

6(b) Substitute $x = 4 \tan \theta$, the answer is $\sqrt{x^2 + 16} + 2 \ln(\sqrt{x^2 + 16} - 4) - 2 \ln(\sqrt{x^2 + 16} + 4) + C$, I think

6(c) Substitute $e^t = 2 \tan \theta$, the answer is $\frac{1}{2} \arctan(\frac{1}{2}e^t) + C$.

6(d) Start with $e^x = \sin \theta$.

6(e) First let $u = \sqrt{x}$, then integrate by parts. Get $2 \cos \sqrt{x} + 2\sqrt{x} \sin \sqrt{x} + C$.

7(a) Answer: 1

7(b) Answer: $\frac{2}{3} \ln 2$

7(c) Answer: $\pi/2$.

9(a) The general solution of the equation is $y = 2 + Ce^{-x}$. For $y(0) = 1$, you need $C = -1$, so the solution of the problem is $y = 2 - e^{-x}$.

12(a) Converges by the ratio test (the ratio is $1/3$).

- 12(b) Diverges by limit comparison to the harmonic series.
- 12(c) Converges by (limit) comparison to the sum of $1/n^2$.
- 12(d) Diverges – the denominator approaches 1 (since e^{-n} approaches zero), but the numerator doesn't approach anything (let alone zero), so it fails the test for divergence (n th term test).
- 12(e) Converges by the ratio test – the ratio is 0.
- 13(a), (c) and (e) – since these series converged without $(-1)^n$, their corresponding alternating versions converge absolutely.
- 13(b) converges conditionally, since these terms approach zero and we already know that the series of absolute values diverges.
- 13(d) Still diverges, since the terms don't approach zero.
- 14(a) Converges by the integral test (same integral as in 7(a) – after a substitution $u = \ln x$ you're integrating $1/u^2$, which converges).
- 14(b) Since $\ln(n!) = \ln 1 + \ln 2 + \cdots + \ln n < 1 + 2 + \cdots + n < n^2$, this series converges by (limit) comparison with the sum of $1/n^2$.
- 14(c) For large n , since $1/n \approx 0$, we have $\tan(1/n) \approx 1/n$. So this series can be limit-compared to the sum of $1/(n \ln n)$, which diverges by the integral test.
- 14(d) The ratio test works to show this one converges (the limit is zero).
- 15(a), (b) and (d) converge absolutely as in problem 13.
- 15(c) Converges conditionally, since the terms decrease and approach zero (alternating series test), but the series of absolute values diverged.
- 16(a) Ratio test gives convergence for $-3 < x < -1$, and at $x = -3$ get the (conditionally convergent) alternating harmonic series, and at $x = -1$ get the (divergent) harmonic series, so interval of convergence is $[-3, 1)$.
- 16(b) Ratio test gives convergence for $2 < x < 4$, and the series converges (absolutely) at both endpoints by integral test or comparison with $\sum 1/n^2$. so interval of convergence is $[2, 4]$.
- 16(c) Ratio test gives convergence for $1 - e < x < 1 + e$, and at $x = 1 + e$ get the alternating harmonic series, at $x = 1 - e$ get the plain harmonic series, so the interval of convergence is $(1 - e, 1 + e]$.
- 16(d) Ratio test gives convergence for $-1 < x < 1$, and at the endpoints, the terms don't approach zero, so the interval of convergence is $(-1, 1)$.
- 17(a) The series in 16(a) is related to the geometric series. Since the integral of $(x+2)^{n-1}$ is $(x+2)^n/n$, the whole series is the integral of $\sum (x+2)^{n-1}$, which is geometric with first

term 1 and ratio $(x + 2)$. So the entire series represents the integral of $1/(1 - (x + 2)) = 1/(-1 - x)$, which is $-\ln(-1 - x)$.

18(a) Substitute $-x^2$ into the series for e^x and get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

18(b) As in part (a), the series for $\cos t^3$ is

$$\cos t^3 = \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n}}{(2n)!}.$$

Then integrate this from 0 to x and get

$$\int_0^x \cos t^3 dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!(6n+1)}.$$

19(a) Since

$$\cos \sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!},$$

the integral is

$$\sum_{n=0}^{\infty} \frac{(-1)^n (0.2)^{n+1}}{(2n)!(n+1)} = 0.2 - \frac{(0.2)^2}{2! \cdot 2} + \frac{(0.2)^3}{4! \cdot 3} - \dots$$

Since this is an alternating series, and the last term shown here is less than 0.001, we can stop after the first two terms, to get

$$\int_0^{0.2} \cos \sqrt{x} dx \approx 0.2 - \frac{0.04}{4} = 0.190$$

to within 0.001.

20. The Maclaurin series for $x^3 \cos x^2$ is $x^3 - \frac{x^7}{2!} + \frac{x^{11}}{4!} - \frac{x^{15}}{6!} - \dots$. Since there's no x^{13} term, $f^{(13)}(0) = 0$.