MATH 371 – Homework assignment 4 – September 20, 2013

1. (a) Find a parametrization for $V(x + 2y - 2z + w + 1\,,\, x + y + z - w - 2)$.

(b) Find the equation of the affine variety determined by the parametric equations

$$x = \frac{t}{1+t}\,, \quad y = 1 - \frac{1}{t^2}.$$

2. (a) Draw a graph in $\mathbb{R}^2$ of the affine variety $V(x^2 - y^2 - 1)$ – this is the "unit hyperbola".

(b) Find a rational parametrization of the hyperbola by considering non-vertical lines through $(-1, 0)$ and their intersections with the hyperbola. Interpret what happens for values of the parameter for which the parametrization is undefined.

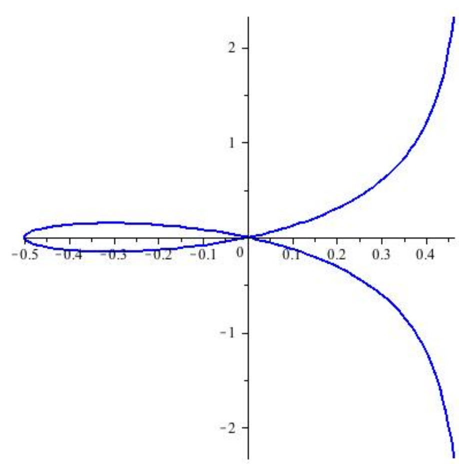
3. (a) Parametrize the sphere $V(x^2 + y^2 + z^2 - 1)$ in $\mathbb{R}^3$ using reasoning similar to the method in the notes for parametrizing the circle. Use lines through the point $(0, 0, 1)$ that intersect the xy-plane at the point $(u, v, 0)$ and then intersect the sphere at the point $x = h_1(u, v)$. $y = h_2(u, v)$, $z = h_3(u, v)$ — your task is to find the three rational functions h_1, h_2 and h_3.

(b) Use similar reasoning to parametrize the $n - 1$-dimensional sphere $V(x_1^2 + \cdots + x_n^2 - 1)$ in $\mathbb{R}^n$. Keep in mind that you should need $n - 1$ parameters.

4. The *strophoid* is the curve given parametrically by

$$x = a \sin t\,, \quad y = a \tan t(1 + \sin t)$$

where a is a constant ($a = \frac{1}{2}$ in the following picture, the curve is asymptotic to the line $x = a$ as $y \to \pm\infty$):



(a) Find a polynomial $p(x, y)$ so that the strophoid is $V(p)$. (Careful, the answer is *not* $(a^2 - x^2)y^2 - x^2(a + x)^2$).

(b) Find an algebraic parametrization of the strophoid.

5. (a) Let $I \subset k[x_1, \ldots, x_n]$ be an ideal, and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_s]$. Explain why the following statements are equivalent:

- $f_1, \ldots, f_s \in I$.
- $\langle f_1, \ldots, \boldsymbol{f}_s \rangle \subset I$.

(b) Use (a) to prove the following pairs of ideals are equal in $\mathbb{Q}[x, y]$:

1. $\langle x + y, x - y \rangle = \langle x, y \rangle$
2. $\langle x + xy, y + xy, x^2, y^2 \rangle = \langle x, y \rangle$
3. $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3 \rangle = \langle x^2 - 4, y^2 - 1 \rangle$

6. (a) Let $V \in \mathbb{R}^3$ be the curve parametrized by (t, t^3, t^4). Prove that V is an affine variety and determine $I(V)$ (use the reasoning given in the notes for the twisted cubic).

(b) Repeat part (a) for the curve parametrized as (t^2, t^3, t^4). This one's a little harder.

7. Let $\mathbb{F}_2$ be the field with two elements (we had been calling this $\mathbb{Z}/2$). Find the ideal $I \subset \mathbb{F}_2[x, y]$ consisting of all polynomials that are zero at every point (all four of them) of $\mathbb{F}_2^2$.

8. The Nullstellensatz is fairly straightforward for the ring $\mathbb{C}[x]$ because every polynomial $p \in \mathbb{C}[x]$ factors completely into linear factors:

$$p(x) = a(x - r_1)^{e_1}(x - r_2)^{e_2} \cdots (x - r_k)^{e_k}.$$

(a) For the polynomial above, what is $V(p)$?

(b) We have $\deg(p) = e_1 + \cdots + e_k$. It is possible that $\deg(p) > k$. Find a polynomial q such that $\deg(q) = k$ and $V(q) = V(p)$. What is $I(V(p))$?

(c) The polynomial q in the above is called the *reduced* or *square-free* part of p. We'll denote it as $q = f_R$. Prove that

$$f_R = \frac{f}{\gcd(f, f')}$$

where f' is the (formal) derivative of f, i.e., if $f = a_m x^m + \cdots + a_0$ then $f' = m a_m x^{m-1} + \cdots + a_1$.

(d) Show that $x^3 + x + 1$ is square-free.

(e) Find the square-free part of

$$x^{11} - x^{10} + 2x^8 - 4x^7 + 3x^5 - 3x^4 + x^3 + 3x^2 - x - 1.$$