MATH 371 – Homework / Exam practice– September 29, 2013

1. (a) Find $\gcd(50+10i\,,\,6+8i)$ in $\mathbb{Z}[i]$, and find Gaussian integers λ and μ such that $(50+10i)\lambda + (6+8i)\mu = \gcd(50+10i\,,\,6+8i)$.

(b) Find a polynomial $p(x) \in \mathbb{R}[x]$ such that

$$\langle p \rangle = \langle x^5 + x^4 + 3x^3 + 3x^2 + 2x + 2\,,\, x^4 - x^3 - x^2 - x - 2 \rangle \subset \mathbb{R}[x],$$

and prove that the two ideals are equal.

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2. (a) An element x of a ring R is called *nilpotent* if $x^n = 0$ for some $n > 0$. Find all the nilpotent elements in the ring $\mathbb{Z}/\langle 36 \rangle$.

(b) Prove that in any commutative ring R, the set of nilpotent elements is an ideal.

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3. A woman goes to market and a horse steps on her basket and crushes her eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same had happened when she picked them out three, four, five and six at a time, but when she took them out seven at a time they came out even (no eggs left over). What is the smallest number of eggs she could have had?

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4. (a) Give an example of a ring R and a prime ideal $P \subset R$ that is not a principal ideal.

(b) Likewise, give an example of a ring R and a principal ideal $\langle x \rangle \subset R$ that is not prime.

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5. Let $R = \mathbb{Z}[i]/\langle 1+3i \rangle$.

(a) Show that $i - 3 \in \langle 1+3i \rangle$ and that $[i] = [3]$ in R (here, $[i]$ is the coset of $\langle 1+3i \rangle$ containing i etc). Use this to prove that $[10] = [0]$ in R and that $[a+bi] = [a+3b]$, where $a, b \in \mathbb{Z}$.

(b) Show that the unique ring homomorphism $\varphi\colon \mathbb{Z} \to R$ (once you've sent 1 to 1, everything else is determined) is surjective.

(c) Show that $1+3i$ is not a unit in $\mathbb{Z}[i]$ and that $1+3i \nmid 2$ and $1+3i \nmid 5$ in $\mathbb{Z}[i]$. Explain why this implies $\ker\varphi = 10\mathbb{Z} = \{10a \mid a \in \mathbb{Z}\}$.

(d) Show that $R \cong \mathbb{Z}/\langle 10 \rangle$.

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6. (a) Consider the curve $y^2 = cx^2 - x^3$ in $\mathbb{R}^2$. Show that this curve is a rational affine variety by finding a rational parametrization using the slopes of lines through the origin (which is a double point for the curve). What happens in $\mathbb{C}^2$?

(b) Show that the four-leaved rose ($r = \sin 2\theta$ in polar coordinates) is an affine variety.