Math 410 **Dr. DeTurck**
Assignment 7 (Last one!) **Due Thursday, December 10, 2009**

Reading: Textbook, bits of chapters 5, 6 and 8 – chapter 5 on infinite products, chapter 6 on the gamma function and chapter 8 on the basics of conformal mapping (not the Riemann mapping theorem).

Problems to hand in:

1. Textbook page 154, problems 6, 8 and 10

2. Some practice with infinite products: Show that

   (a) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$

   (b) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right)=\frac{1}{3}$

   (c) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{2}{n^3+1}\right)=\frac{2}{3}$

   (Part of the problem in each case is to verify convergence!)

3. Recall that
$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$
   where γ is Euler's constant
$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right).$$
   Prove Euler's formula:
$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}$$
   as follows: Use the definition of $1/\Gamma(z)$ given above, except write the infinite product as a limit of partial products. Now, you will have that both γ and the infinite product are limits as $N\to\infty$. Put the two limits together, and somewhere along the way notice that
$$\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)=n$$
   to reach the conclusion.

4. (This is also textbook p. 174, problem 1): Prove Euler's other formula:
$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}.$$
   (Hint: This follows pretty directly by considering the partial products of the previous problem – how's that for alliteration?)

5. Now prove that

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

as follows:

Let $G(z) = \displaystyle\int_0^\infty t^{z-1}e^{-t}\,dt$.

(a) If $F_n(z) = \displaystyle\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,dt$, make the change of variables $s = t/n$ and integrate by parts repeatedly to show that

$$F_n(z) = \frac{n!n^z}{z(z+1)\cdots(z+n)}$$

so that by the previous problem, $\lim_{n\to\infty} F_n(z) = \Gamma(z)$.

(b) Now we'd like to use $\displaystyle\lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n = e^{-t}$ to show that $\Gamma(z) = G(z)$, but we have to be careful about the n in the upper limit of $F_n(z)$ as well as the n in the integrand.

Note that

$$G(z) - \Gamma(z) = \lim_{n\to\infty}\left(\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt + \int_n^\infty e^{-t}t^{z-1}\,dt\right).$$

(i) Explain why the second integral approaches zero as $n \to \infty$.

(ii) Prove that if $0 \le t \le n$, then

$$0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le \frac{t^2e^{-t}}{n}.$$

(iii) Use (ii) to show

$$\left|\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt\right| \le \int_0^n \frac{e^{-t}t^{\operatorname{Re} z+1}}{n}\,dt \le \frac{1}{n}\int_0^\infty e^{-t}t^{\operatorname{Re} z+1}\,dt$$

which approaches zero as $n \to \infty$ provided $\operatorname{Re} z > 0$, because the last integral is finite (before you divide it by n).

6. Out-Legendre Legendre: Show that

$$\Gamma(3z) = \frac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma(z+1/3)\Gamma(z+2/3).$$

(Hint: What are the poles of $\Gamma(3z)$?)

7. If z_0 is a critical point of $f(z)$, and m is the smallest integer such that $f^{(m)}(z_0) \neq 0$, then the mapping given by f multiples angles at z_0 by a factor of m. (Hint: Taylor's theorem!)

8. Find the streamlines for the flow in a wedge of angle $\pi/4$ if the flow comes in along one ray and goes out along the other.

9. Textbook page 249 problem 8