Math 410 **Dr. DeTurck**
Hints and Answers for Practice Problems for Midterm 1 **October 9, 2009**

1. For which values of z is $z^2 = |z|^2$? For which values of z is $z^2 = i|z|^2$?

   If $z^2 = z\overline{z}$, either $z = 0$ or else $z = \overline{z}$. In other words, z is real. For $z^2 = i|z|^2$, write $z = re^{i\theta}$ in polar form, and get that either $r = 0$ or else $e^{i\theta} = e^{i(\pi/2-\theta)}$, so $\theta = \pi/4 + k\pi$, in other words the real and imaginary parts of z must be equal.

2. Let $f(z) = z + 1/z$. What is the image of the unit circle under the mapping defined by f?

   The unit circle is $z = e^{i\theta}$, and $f(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta$. Since θ is real, we get that the image of the unit circle is the *real* interval $[-2, 2]$.

3. On the domain $\{z = x + iy,\ 0 \le x \le 2\pi,\ 0 \le y \le 2\pi\}$, what is the maximum value of $|\cos z|$?

   A computation shows that $|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$. The only critical points of this function on the square in question all have $y = 0$, so we need only look on the boundary of the square.

   So let $\phi = |\cos z|^2$. When $y = 0$, $\phi = \cos^2 x$ so the max is 1. If $x = 0$ or $x = 2\pi$, then $\phi = \cosh^2 y$, and the max occurs at $y = 2\pi$ and is $\cosh^2(2\pi)$. And since $\sinh^2 2\pi < \cosh^2 2\pi$, ϕ never gets any bigger than this on the line $y = 2\pi$. Thus the max of $|\cos z|$ is $\cosh 2\pi$, which occurs at $z = 2\pi i,\ \pi + 2\pi i,\ 2\pi + 2\pi i$.

4. Let $u(x, y) = 2x - xy$. Find a function $v(x, y)$ so that

   $$f(x + iy) = u(x, y) + iv(x, y)$$

   is a holomorphic function. Express $f(z)$ in terms of z alone.

   Need $v_y = u_x = 2-y$, so $v = 2y-y^2/2+f(x)$. Also need $v_x = -u_y = x$, so $f(x) = x^2/2+c$. Thus (up to adding a constant)

   $$f = 2x - xy + i(x^2/2 - y^2/2 + 2y) = 2z - iz^2/2.$$

5. Find all the solutions of $\sin z = \sqrt{3}$.

   This is the same as $q - 1/q = 2i\sqrt{3}$, where $q = e^{iz}$. Solve the resulting quadratic equation and get $q = i(\sqrt{3} \pm 2)$. So

   $$z = -i \ln q = (\pi/2 + 2k\pi) - i \ln(2 + \sqrt{3})$$

   or

   $$z = -i \ln q = (-\pi/2 + 2k\pi) - i \ln(2 - \sqrt{3}).$$

6. Calculate $\int_\gamma \overline{z}\, dz$, $\int_\gamma \frac{dz}{\overline{z}}$, where γ is the unit circle, traversed once in the counterclockwise direction.

   Let $z = e^{i\theta}$, note $\overline{z} = e^{-i\theta}$ and $dz = ie^{i\theta}d\theta$. Substitute, integrate and get

   $$\int_\gamma \overline{z}\, dz = 2\pi i$$

   and

   $$\int_\gamma \frac{dz}{\overline{z}} = 0.$$

7. Give an example of a (nontrivial) simple closed curve γ for which

$$\int_\gamma \frac{dz}{z^2+z+1} = 0$$

and another for which

$$\int_\gamma \frac{dz}{z^2+z+1} \neq 0.$$

What is the value of the second integral over your curve?

The roots of $z^2+z+1=0$ are $z=-1/2 \pm i\sqrt{3}/2$, so use partial fractions to get

$$\frac{1}{z^2+z+1} = \frac{\frac{1}{i\sqrt{3}}}{z+\frac{1}{2}-i\frac{\sqrt{3}}{2}} - \frac{\frac{1}{i\sqrt{3}}}{z+\frac{1}{2}+i\frac{\sqrt{3}}{2}}.$$

So if the curve doesn't enclose either singularity (or in fact if it encloses *both* of them) then the integral is 0 (e.g., $|z-20|=1$ or $|z|=10$).

If the curve encloses $-1/2+i\sqrt{3}/2$ (but not $-1/2-i\sqrt{3}/2$) then the value of the integral is $2\pi i(1/(i\sqrt{3})) = 2\pi/\sqrt{3}$. For instance $|z+1/2-i\sqrt{3}/2| = \sqrt{3}/2$.

8. Calculate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx$$

by applying the Cauchy Integral Formula to

$$\int_\gamma \frac{e^{iz}}{(z+i)(z-i)}\, dz$$

where γ is the "standard" semicircular contour of radius R and letting R go to infinity. Be sure to estimate what happens on the circle part carefully.

You can use $f(z) = e^{iz}/(z+i)$ around the standard contour, and get that the integral of $e^{iz}/(z^2+1)$ equals $2\pi i f(i) = \pi/e$. If we can show that the integral over the curved part goes to zero, then the improper integral (being the real part of the complex one), is also π/e.

To show the integral over the curved part goes to zero, let $z = Re^{i\theta}$, and after parametrizing and substituting, we get the integral is less than the integral:

$$\int_0^\pi \frac{R}{R^2-1}\, d\theta,$$

which certainly goes to zero as $R \to \infty$.