Math 410 **Dr. DeTurck**
Practice Problems for Midterm 2 **November 10, 2009**

1. From Taylor's theorem, we know that if $f(z)$ is holomorphic in an open set Ω that contains the point z_0, and if $f(z_0) = 0$, then we can write $f(z) = (z - z_0)^n q(z)$ for some integer $n > 0$ and a function $q(z)$ that is holomorphic in an open subset $U \subset \Omega$ that contains z_0, and $q(z_0) \neq 0$ (and then we say that f has a zero of order n at z_0). Use this to prove the following version of L'Hôpital's rule:

   If f and g are holomorphic in a neighborhood of z_0, and if $f(z_0) = g(z_0) = 0$ and if $g'(z_0) \neq 0$, then
   $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$

2. Calculate $\displaystyle\int_{|z|=1} \frac{\sin z}{z^4}\, dz$. (Hint: Use the series for the sine function to get the residue).

3. Find the maximum of $|e^z|$ on the disk $|z| \leq 1$ (think first!).

4. Calculate the integral $\displaystyle\int_0^\infty \frac{1}{x^3+1}\, dx$ using integration over two-thirds of the standard semi-circular contour (i.e., instead of coming back to the origin along the negative real axis – where there is a pole at $x = -1$ – come back along the line where the argument of z is $2\pi/3$).

5. (For Stefan) Calculate $\displaystyle\int_0^\pi \sin^{2n}\theta\, d\theta$. (At some point you will need the binomial theorem.)

6. Suppose f is an entire function and $|f(z)| \leq 10\sqrt{|z|}$ for all z such that $|z| > 1$. Must f be constant? Prove it or explain why not.

7. Show that if f is a meromorphic function that has no poles at integers, and $\displaystyle\lim_{|z|\to\infty} |zf(z)| = 0$, then
   $$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum R_{z_k},$$
   where z_k are the poles of f and R_k is the residue of the function $\pi f(z)/\sin \pi z$ at z_k.

   Use this result to show that
   $$\sum_{n=\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{a \sinh \pi a}.$$