Math 410 **Dr. DeTurck**
Hints and Answers to Practice Problems for Midterm 2 **November 10, 2009**

1. From Taylor's theorem, we know that if $f(z)$ is holomorphic in an open set Ω that contains the point z_0, and if $f(z_0) = 0$, then we can write $f(z) = (z - z_0)^n q(z)$ for some integer $n > 0$ and a function $q(z)$ that is holomorphic in an open subset $U \subset \Omega$ that contains z_0, and $q(z_0) \neq 0$ (and then we say that f has a zero of order n at z_0). Use this to prove the following version of L'Hôpital's rule:

   If f and g are holomorphic in a neighborhood of z_0, and if $f(z_0) = g(z_0) = 0$ and if $g'(z_0) \neq 0$, then
   $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$

   Since $g'(z_0) \neq 0$, we can write $g(z) = (z - z_0)r(z)$ where $f(z_0) \neq 0$. Thus:
   $$\frac{f(z)}{g(z)} = \frac{(z - z_0)^n q(z)}{(z - z_0)r(z)} = \frac{(z - z_0)^{n-1} q(z)}{r(z)}$$
   so if $n > 1$ then the limit is zero and if $n = 1$ then the limit is $q(z_0)/r(z_0)$.

   Likewise
   $$\frac{f'(z)}{g'(z)} = \frac{n(z - z_0)^{n-1} q(z) - (z - z_0)q'(z)}{r(z) + (z - z_0)r'(z)},$$
   and again if $n > 1$ the limit is zero and if $n = 1$ then the limit is $q(z_0)/r(z_0)$, which proves the result.

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2. Calculate $\displaystyle\int_{|z|=1} \frac{\sin z}{z^4}\, dz$. (Hint: Use the series for the sine function to get the residue).

   Use the residue theorem, the only singularity is at $z = 0$. To calculate the residue there, recall that
   $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$
   so that
   $$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \cdots.$$
   From this we see that the residue of $\sin z/z^4$ is $-1/6$. Therefore the value of the integral is $-2\pi i/6 = -\pi i/3$.

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3. Find the maximum of $|e^z|$ on the disk $|z| \leq 1$ (think first!).

   From the maximum principle, we know that the maximum occurs on the boundary of the disk, where $|z| = 1$. There, $z = \cos t + i \sin t$, and so
   $$|e^z| = |e^{\cos t + i \sin t}| = |e^{\cos t}|.$$
   Thus, the maximum is e (since the max of $\cos t$ is 1).

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4. Calculate the integral $\int_0^\infty \frac{1}{x^3+1}\,dx$ using integration over two-thirds of the standard semi-circular contour (i.e., instead of coming back to the origin along the negative real axis – where there is a pole at $x=-1$ – come back along the line where the argument of z is $2\pi/3$).

The poles of $1/(z^3+1)$ are at $e^{i\pi/3}$, -1 and $e^{i5\pi/3}$, so the only pole inside the contour is $\alpha = e^{i\pi/3}$ and it's a simple pole. The residue at this pole is

$$\lim_{z\to\alpha} \frac{z-\alpha}{z^3+1} = \lim_{z\to\alpha} \frac{1}{3z^2} = \frac{1}{3\alpha^2} = \frac{1}{3e^{2\pi i/3}}$$

using L'Hôpital's rule.

The integral over the circular part of the contour is seen to go to zero, since along it, $z = Re^{it}$, and so it is estimated by

$$\left| \int_0^{2\pi/3} \frac{iRe^{it}dt}{R^3e^{3it}+1} \right| \le \frac{(2\pi/3)R^2}{R^3-1},$$

the limit of which is zero as $R\to\infty$.

Finally, along the return line, $z = re^{2\pi i/3}$, $dz = e^{2\pi i/3}dr$, and $z^3 = r^3$ (since $e^{2\pi i} = 1$. Thus, this part of the integral is

$$\int_\infty^0 \frac{e^{2\pi i/3}dr}{r^3+1} = -e^{2\pi i/3}\int_0^\infty \frac{dr}{1+r^3}.$$

Therefore,

$$\frac{2\pi i}{3e^{2\pi i/3}} = (1-e^{2\pi i/3})\int_0^\infty \frac{dx}{x^3+1},$$

and so

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi i}{3(1-e^{2\pi i/3})e^{2\pi i/3}} = \frac{2\pi i}{3(e^{2\pi i/3}-e^{4\pi i/3})} = \frac{2\pi i}{3\sqrt{3}i} = \frac{2\pi}{3\sqrt{3}}.$$

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5. (For Stefan) Calculate $\int_0^\pi \sin^{2n}\theta\,d\theta$. (At some point you will need the binomial theorem.)

Let $z = e^{i\theta}$, so $d\theta = dz/(iz)$, and

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e{-}i\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Thus

$$\int_0^\pi \sin^{2n}\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}\sin^{2n}\theta\,d\theta = \int_{|z|=1}\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^{2n}\frac{dz}{iz}.$$

The integrand has a pole at $z=0$. To calculate its residue there, we expand the integrand in powers of z, and the residue is the coefficient of $1/z$: The integrand is:

$$\frac{1}{(2i)^{2n}iz}\sum_{k=0}^{2n}(-1)^k C_k^{2n}\frac{z^k}{z^{2n-k}},$$

where C_k^{2n} is the binomial coefficient

$$C_k^{2n} = \frac{(2n)!}{k!(2n-k)!}.$$

Because of the z in the denominator in front of the sum, we need the constant term (where $k = 2n - k$, or $k = n$) in the sum. Therefore, the residue is

$$\frac{1}{(2i)^{2n} i} (-1)^n C_n^{2n} = \frac{C_n^{2n}}{2^{2n} i}.$$

We have to multiply this by πi (not $2\pi i$, because of the $1/2$), and so the answer is

$$\frac{\pi C_n^{2n}}{2^{2n}}.$$

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6. Suppose f is an entire function and $|f(z)| \leq 10\sqrt{|z|}$ for all z such that $|z| > 1$. Must f be constant? Prove it or explain why not.

   The function must be constant. For any z_0, use Cauchy's integral formula around a large circle about z_0 to write

   $$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\, dz \right|.$$

   And we can estimate this by $1/\sqrt{R}$, by hypothesis, so $f'(z_0) = 0$. Since z_0 was aribtrary, f is constant.

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7. Show that if f is a meromorphic function that has no poles at integers, and $\lim_{|z|\to\infty} |zf(z)| = 0$, then

   $$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum R_{z_k},$$

   where z_k are the poles of f and R_k is the residue of the function $\pi f(z)/\sin \pi z$ at z_k.

   Use this result to show that

   $$\sum_{n=\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{a \sinh \pi a}.$$

   Done in class; maybe later.