17.1 Complex Numbers

You have undoubtedly encountered complex numbers in your earlier mathematics classes. From the quadratic formula we know that the zeros of the quadratic function $f(x) = ax^2 + bx + c$ are complex whenever the discriminant $b^2 - 4ac$ is negative. Simple equations such as $x^2 = -5$ and $x^2 + x + 1 = 0$ have no real-number solutions.

DEFINITION 17.1 Complex Number

A number of the form $z = x + iy$ where x and y are real numbers and i is a number such that $i^2 = -1$ is called a **complex number.**

Terminology The number i in Definition 17.1 is called the **imaginary unit.** The real number x in $z = x + iy$ is called the **real part** of z; the real number y is called the **imaginary part** of z. The real and imaginary parts of a complex number z are abbreviated Re(z) and Im(z), respectively. For example, if $z = 4 - 9i$, then Re(z) = 4 and Im(z) = −9. A real constant multiple of the imaginary unit is called a **pure imaginary number.** For example, $z = 6i$ is a pure imaginary number. Two complex numbers are **equal** if their real and imaginary parts are equal. Since this simple concept is sometimes useful, we formalize the last statement in the next definition.

DEFINITION 17.2 Equality

Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are **equal,** $z_1 = z_2$, if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

A complex number $x + iy = 0$ if $x = 0$ and $y = 0$.

Arithmetic Operations Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, these operations are defined as follows.

Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

Multiplication:
$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2)$$

Division:
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

The familiar commutative, associative, and distributive laws hold for complex numbers.

Commutative laws:
$$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1z_2 = z_2z_1 \end{cases}$$

Associative laws:
$$\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2z_3) = (z_1z_2)z_3 \end{cases}$$

Distributive law: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

In view of these laws, there is no need to memorize the definitions of addition, subtraction, and multiplication. To add (subtract) two complex numbers, we simply add (subtract) the corresponding real and imaginary parts. To multiply two complex numbers, we use the distributive law and the fact that $i^2 = -1$.

Example 1 **Addition and Multiplication**

If $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$, find (a) $z_1 + z_2$ and (b) $z_1 z_2$.

SOLUTION (a) By adding the real and imaginary parts of the two numbers, we get

$$(2 + 4i) + (-3 + 8i) = (2 - 3) + (4 + 8)i = -1 + 12i.$$

(b) Using the distributive law, we have

$$\begin{aligned}(2 + 4i)(-3 + 8i) &= (2 + 4i)(-3) + (2 + 4i)(8i)\\ &= -6 - 12i + 16i + 32i^2\\ &= (-6 - 32) + (16 - 12)i = -38 + 4i.\end{aligned}$$ ❑

There is also no need to memorize the definition of division, but before discussing that we need to introduce another concept.

Conjugate

If z is a complex number, then the number obtained by changing the sign of its imaginary part is called the **complex conjugate** or, simply, the **conjugate** of z. If $z = x + iy$, then its conjugate is

$$\bar{z} = x - iy.$$

For example, if $z = 6 + 3i$, then $\bar{z} = 6 - 3i$; if $z = -5 - i$, then $\bar{z} = -5 + i$. If z is a real number, say $z = 7$, then $\bar{z} = 7$. From the definition of addition it can be readily shown that the conjugate of a sum of two complex numbers is the sum of the conjugates:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$$

Moreover, we have the additional three properties

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \qquad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

The definitions of addition and multiplication show that the sum and product of a complex number z and its conjugate $\bar{z}$ are also real numbers:

$$z + \bar{z} = (x + iy) + (x - iy) = 2x \tag{1}$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2. \tag{2}$$

The difference between a complex number z and its conjugate $\bar{z}$ is a pure imaginary number:

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy. \tag{3}$$

Since $x = \text{Re}(z)$ and $y = \text{Im}(z)$, (1) and (3) yield two useful formulas:

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

However, (2) is the important relationship that enables us to approach division in a more practical manner: To divide z_1 by z_2, we multiply both numerator and denominator of z_1/z_2 by the conjugate of z_2. This will be illustrated in the next example.

Example 2 **Division**

If $z_1 = 2 - 3i$ and $z_2 = 4 + 6i$, find (a) $\dfrac{z_1}{z_2}$ and (b) $\dfrac{1}{z_1}$.

SOLUTION In both parts of this example we shall multiply both numerator and denominator by the conjugate of the denominator and then use (2).

(a) $$\frac{2-3i}{4+6i} = \frac{2-3i}{4+6i}\,\frac{4-6i}{4-6i} = \frac{8-12i-12i+18i^2}{16+36}$$
$$= \frac{-10-24i}{52} = -\frac{5}{26} - \frac{6}{13}\,i.$$

(b) $$\frac{1}{2-3i} = \frac{1}{2-3i}\,\frac{2+3i}{2+3i} = \frac{2+3i}{4+9} = \frac{2}{13} + \frac{3}{13}\,i.$$ ❑

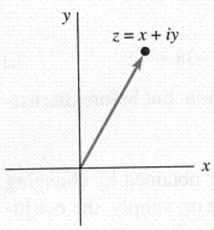


Figure 17.1

■ Geometric Interpretation

A complex number $z = x + iy$ is uniquely determined by an *ordered pair* of real numbers (x, y). The first and second entries of the ordered pairs correspond, in turn, with the real and imaginary parts of the complex number. For example, the ordered pair $(2, -3)$ corresponds to the complex number $z = 2 - 3i$. Conversely, $z = 2 - 3i$ determines the ordered pair $(2, -3)$. In this manner we are able to associate a complex number $z = x + iy$ with a point (x, y) in a coordinate plane. But, as we saw in Section 7.1, an ordered pair of real numbers can be interpreted as the components of a vector. Thus, a complex number $z = x + iy$ can also be viewed as a *vector* whose initial point is the origin and whose terminal point is (x, y). The coordinate plane illustrated in Figure 17.1 is called the **complex plane** or simply the ***z*-plane.** The horizontal or x-axis is called the **real axis** and the vertical or y-axis is called the **imaginary axis.** The length of a vector z, or the distance from the origin to the point (x, y), is clearly $\sqrt{x^2 + y^2}$. This real number is given a special name.

DEFINITION 17.3 Modulus or Absolute Value

The **modulus** or **absolute value** of $z = x + iy$, denoted by $|z|$, is the real number
$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{4}$$

Example 3 **Modulus of a Complex Number**

If $z = 2 - 3i$, then $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. ❑

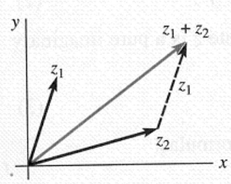


Figure 17.2

As Figure 17.2 shows, the sum of the vectors z_1 and z_2 is the vector $z_1 + z_2$. For the triangle given in the figure we know that the length of the side of the triangle corresponding to the vector $z_1 + z_2$ cannot be longer than the sum of the remaining two sides. In symbols this is
$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{5}$$

The result in (5) is known as the **triangle inequality** and extends to any finite sum:
$$|z_1 + z_2 + z_3 + \cdots + z_n| \le |z_1| + |z_2| + |z_3| + \cdots + |z_n|. \tag{6}$$

Using (5) on $z_1 + z_2 + (-z_2)$, we obtain another important inequality:
$$|z_1 + z_2| \ge |z_1| - |z_2|. \tag{7}$$

Remarks

Many of the properties of the real system hold in the complex number system, but there are some remarkable differences as well. For example, we cannot compare two complex numbers $z_1 = x_1 + iy_1$, $y_1 \neq 0$, and $z_2 = x_2 + iy_2$, $y_2 \neq 0$, by means of inequalities. In other words, statements such as $z_1 < z_2$ and $z_2 \geq z_1$ have no meaning except in the case when the two numbers z_1 and z_2 are real. We can, however, compare the absolute values of two complex numbers. Thus, if $z_1 = 3 + 4i$ and $z_2 = 5 - i$, then $|z_1| = 5$ and $|z_2| = \sqrt{26}$, and consequently $|z_1| < |z_2|$. This last inequality means that the point (3, 4) is closer to the origin than is the point (5, −1).

EXERCISES 17.1

Answers to odd-numbered problems begin on page A-71.

In Problems 1–26, write the given number in the form $a + ib$.

1. $2i^3 - 3i^2 + 5i$
2. $3i^5 - i^4 + 7i^3 - 10i^2 - 9$
3. i^8
4. i^{11}
5. $(5 - 9i) + (2 - 4i)$
6. $3(4 - i) - 3(5 + 2i)$
7. $i(5 + 7i)$
8. $i(4 - i) + 4i(1 + 2i)$
9. $(2 - 3i)(4 + i)$
10. $(\frac{1}{2} - \frac{1}{4}i)(\frac{2}{3} + \frac{5}{3}i)$
11. $(2 + 3i)^2$
12. $(1 - i)^3$
13. $\dfrac{2}{i}$
14. $\dfrac{i}{1 + i}$
15. $\dfrac{2 - 4i}{3 + 5i}$
16. $\dfrac{10 - 5i}{6 + 2i}$
17. $\dfrac{(3 - i)(2 + 3i)}{1 + i}$
18. $\dfrac{(1 + i)(1 - 2i)}{(2 + i)(4 - 3i)}$
19. $\dfrac{(5 - 4i) - (3 + 7i)}{(4 + 2i) + (2 - 3i)}$
20. $\dfrac{(4 + 5i) + 2i^3}{(2 + i)^2}$
21. $i(1 - i)(2 - i)(2 + 6i)$
22. $(1 + i)^2(1 - i)^3$
23. $(3 + 6i) + (4 - i)(3 + 5i) + \dfrac{1}{2 - i}$
24. $(2 + 3i)\left(\dfrac{2 - i}{1 + 2i}\right)^2$
25. $\left(\dfrac{i}{3 - i}\right)\left(\dfrac{1}{2 + 3i}\right)$
26. $\dfrac{1}{(1 + i)(1 - 2i)(1 + 3i)}$

In Problems 27–32, let $z = x + iy$. Find the indicated expression.

27. $\text{Re}(1/z)$
28. $\text{Re}(z^2)$
29. $\text{Im}(2z + 4\bar{z} - 4i)$
30. $\text{Im}(\bar{z}^2 + z^2)$
31. $|z - 1 - 3i|$
32. $|z + 5\bar{z}|$

In Problems 33–36, use Definition 17.2 to find a complex number z satisfying the given equation.

33. $2z = i(2 + 9i)$
34. $z - 2\bar{z} + 7 - 6i = 0$
35. $z^2 = i$
36. $\bar{z}^2 = 4z$

In Problems 37 and 38, determine which complex number is closer to the origin.

37. $10 + 8i$, $11 - 6i$
38. $\frac{1}{2} - \frac{1}{4}i$, $\frac{2}{3} + \frac{1}{6}i$
39. Prove that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 in the complex plane.
40. Show for all complex numbers z on the circle $x^2 + y^2 = 4$ that $|z + 6 + 8i| \leq 12$.

17.2 Polar Form of Complex Numbers; Powers and Roots

■ Polar Form Recall from calculus that a point (x, y) in rectangular coordinates has the polar description (r, θ), where x, y, r, and θ are related by $x = r\cos\theta$ and $y = r\sin\theta$. Thus a nonzero complex number $z = x + iy$ can be written as $z = (r\cos\theta) + i(r\sin\theta)$ or

$$z = r(\cos\theta + i\sin\theta). \quad (1)$$

We say that (1) is the **polar form** of the complex number z. We see from Figure 17.3 that the coordinate r can be interpreted as the distance from the origin to the

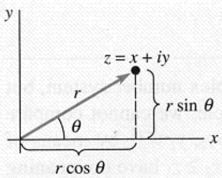


Figure 17.3

point (x, y). In other words, we adopt the convention that r is never negative so that we can take r to be the modulus of z, that is, $r = |z|$. The angle θ of inclination of the vector z measured in radians from the positive real axis is positive when measured counterclockwise and negative when measured clockwise. The angle θ is called an **argument** of z and is written $\theta = \arg z$. From Figure 17.3 we see that an argument of a complex number must satisfy the equation $\tan\theta = y/x$. The solutions of this equation are not unique, since if θ_0 is an argument of z, then necessarily the angles $\theta_0 \pm 2\pi, \theta_0 \pm 4\pi, \ldots$, are also arguments. The argument of a complex number in the interval $-\pi < \theta \le \pi$ is called the **principal argument** of z and is denoted by $\operatorname{Arg} z$. For example, $\operatorname{Arg}(i) = \pi/2$.

Example 1 **A Complex Number in Polar Form**

Express $1 - \sqrt{3}i$ in polar form.

SOLUTION With $x = 1$ and $y = -\sqrt{3}$, we obtain $r = |z| = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$. Now since the point $(1, -\sqrt{3})$ lies in the fourth quadrant, we can take the solution of $\tan\theta = -\sqrt{3}/1 = -\sqrt{3}$ to be $\theta = \arg z = 5\pi/3$. It follows from (1) that a polar form of the number is

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right).$$

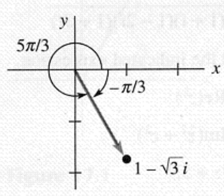


Figure 17.4

As we see in Figure 17.4, the argument of $1 - \sqrt{3}i$ that lies in the interval $(-\pi, \pi]$, the principal argument of z, is $\operatorname{Arg} z = -\pi/3$. Thus, an alternative polar form of the complex number is

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right].$$ □

■ Multiplication and Division

The polar form of a complex number is especially convenient to use when multiplying or dividing two complex numbers. Suppose

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

where θ_1 and θ_2 are any arguments of z_1 and z_2, respectively. Then

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \quad (2)$$

and for $z_2 \ne 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]. \quad (3)$$

From the addition formulas from trigonometry, (2) and (3) can be rewritten, in turn, as

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \quad (4)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \quad (5)$$

Inspection of (4) and (5) shows that

$$|z_1z_2| = |z_1|\,|z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad (6)$$

and

$$\arg(z_1z_2) = \arg z_1 + \arg z_2, \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \quad (7)$$

Example 2 **Argument of a Product and of a Quotient**

We have seen that Arg $z_1 = \pi/2$ for $z_1 = i$. In Example 1 we saw that Arg $z_2 = -\pi/3$ for $z_2 = 1 - \sqrt{3}i$. Thus, for

$$z_1 z_2 = i(1 - \sqrt{3}i) = \sqrt{3} + i \quad \text{and} \quad \frac{z_1}{z_2} = \frac{1 - \sqrt{3}i}{i} = -\sqrt{3} - i$$

it follows from (7) that

$$\arg(z_1 z_2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} - \left(-\frac{\pi}{3}\right) = \frac{5\pi}{6}. \quad \square$$

In Example 2 we used the principal arguments of z_1 and z_2 and obtained $\arg(z_1 z_2) = \operatorname{Arg}(z_1 z_2)$ and $\arg(z_1/z_2) = \operatorname{Arg}(z_1/z_2)$. It should be observed, however, that this was a coincidence. Although (7) is true for any arguments of z_1 and z_2, *it is not true,* in general, that $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ and $\operatorname{Arg}(z_1/z_2) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$. See Problem 39 in Exercises 17.2.

Powers of z

We can find integer powers of the complex number z from the results in (4) and (5). For example, if $z = r(\cos\theta + i\sin\theta)$, then with $z_1 = z$ and $z_2 = z$, (4) gives

$$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] = r^2(\cos 2\theta + i\sin 2\theta).$$

Since $z^3 = z^2 z$, it follows that

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta).$$

Moreover, since $\arg(1) = 0$, it follows from (5) that

$$\frac{1}{z^2} = z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)].$$

Continuing in this manner, we obtain a formula for the nth power of z for any integer n:

$$z^n = r^n(\cos n\theta + i\sin n\theta). \tag{8}$$

Example 3 **Power of a Complex Number**

Compute z^3 for $z = 1 - \sqrt{3}i$.

SOLUTION In Example 1 we saw that

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right].$$

Hence from (8) with $r = 2$, $\theta = -\pi/3$, and $n = 3$, we get

$$(1 - \sqrt{3}i)^3 = 2^3\left[\cos\left(3\left(-\frac{\pi}{3}\right)\right) + i\sin\left(3\left(-\frac{\pi}{3}\right)\right)\right]$$

$$= 8[\cos(-\pi) + i\sin(-\pi)] = -8. \quad \square$$

DeMoivre's Formula

When $z = \cos\theta + i\sin\theta$, we have $|z| = r = 1$ and so (8) yields

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{9}$$

This last result is known as **DeMoivre's formula** and is useful in deriving certain trigonometric identities.

■ Roots A number w is said to be an ***n*th root** of a nonzero complex number z if $w^n = z$. If we let $w = \rho(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$ be the polar forms of w and z, then in view of (8) $w^n = z$ becomes

$$\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta).$$

From this we conclude that $\rho^n = r$ or $\rho = r^{1/n}$ and

$$\cos n\phi + i\sin n\phi = \cos\theta + i\sin\theta.$$

By equating the real and imaginary parts, we get from this equation

$$\cos n\phi = \cos\theta \quad \text{and} \quad \sin n\phi = \sin\theta.$$

These equalities imply that $n\phi = \theta + 2k\pi$, where k is an integer. Thus,

$$\phi = \frac{\theta + 2k\pi}{n}.$$

As k takes on the successive integer values $k = 0, 1, 2, \ldots, n-1$, we obtain n *distinct* roots with the same modulus but different arguments. But for $k \geq n$ we obtain the same roots because the sine and cosine are 2π-periodic. To see this, suppose $k = n + m$, where $m = 0, 1, 2, \ldots$. Then

$$\phi = \frac{\theta + 2(n+m)\pi}{n} = \frac{\theta + 2m\pi}{n} + 2\pi$$

and so $\quad \sin\phi = \sin\left(\frac{\theta + 2m\pi}{n}\right), \qquad \cos\phi = \cos\left(\frac{\theta + 2m\pi}{n}\right).$

We summarize this result. The nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], \tag{10}$$

where $k = 0, 1, 2, \ldots, n-1$.

Example 4 Roots of a Complex Number

Find the three cube roots of $z = i$.

SOLUTION With $r = 1$, $\theta = \arg z = \pi/2$, the polar form of the given number is $z = \cos(\pi/2) + i\sin(\pi/2)$. From (10) with $n = 3$ we obtain

$$w_k = (1)^{1/3}\left[\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right)\right], \qquad k = 0, 1, 2.$$

Hence, the three roots are:

$$k = 0, \qquad w_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 1, \qquad w_1 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 2, \qquad w_2 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i.$$

❑

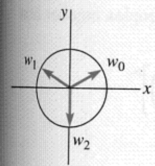


Figure 17.5

The root w of a complex number z obtained by using the principal argument of z with $k = 0$ is sometimes called the **principal nth root** of z. In Example 4, since Arg $(i) = \pi/2$, $w_0 = (\sqrt{3}/2) + (1/2)i$ is the principal third root of i.

Since the roots given by (8) have the same modulus, the n roots of a nonzero complex number z lie on a circle of radius $r^{1/n}$ centered at the origin in the complex plane. Moreover, since the difference between the arguments of any two successive roots is $2\pi/n$, the nth roots of z are equally spaced on this circle. Figure 17.5 shows the three roots of i equally spaced on a unit circle; the angle between roots (vectors) w_k and w_{k+1} is $2\pi/3$.

As the next example will show, the roots of a complex number do not have to be "nice" numbers as in Example 3.

Example 5 Roots of a Complex Number

Find the four fourth roots of $z = 1 + i$.

SOLUTION In this case, $r = \sqrt{2}$ and $\theta = \arg z = \pi/4$. From (10) with $n = 4$ we obtain

$$w_k = (2)^{1/4}\left[\cos\left(\frac{\pi/4 + 2k\pi}{4}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{4}\right)\right], \qquad k = 0, 1, 2, 3.$$

Thus,

$$k = 0, \qquad w_0 = (2)^{1/4}\left[\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right] = 1.1664 + 0.2320i$$

$$k = 1, \qquad w_1 = (2)^{1/4}\left[\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right] = -0.2320 + 1.1664i$$

$$k = 2, \qquad w_2 = (2)^{1/4}\left[\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right] = -1.1664 - 0.2320i$$

$$k = 3, \qquad w_3 = (2)^{1/4}\left[\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right] = 0.2320 - 1.1664i. \quad \square$$

EXERCISES 17.2

Answers to odd-numbered problems begin on page A-71.

In Problems 1–10, write the given complex number in polar form.

1. 2 **2.** −10

3. $-3i$ **4.** $6i$

5. $1 + i$ **6.** $5 - 5i$

7. $-\sqrt{3} + i$ **8.** $-2 - 2\sqrt{3}i$

9. $\dfrac{3}{-1 + i}$ **10.** $\dfrac{12}{\sqrt{3} + i}$

In Problems 11–14, write the number given in polar form in the form $a + ib$.

11. $z = 5\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

12. $z = 8\sqrt{2}\left(\cos\frac{11\pi}{4} + i\sin\frac{11\pi}{4}\right)$

13. $z = 6\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

14. $z = 10\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

In Problems 15 and 16, find z_1z_2 and z_1/z_2. Write the number in the form $a + ib$.

15. $z_1 = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$, $z_2 = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$

16. $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, $z_2 = \sqrt{3}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

In Problems 17–20, write each complex number in polar form. Then use either (4) or (5) to obtain a polar form of the given number. Write the polar form in the form $a + ib$.

17. $(3 - 3i)(5 + 5\sqrt{3}i)$ **18.** $(4 + 4i)(-1 + i)$

19. $\dfrac{-i}{2-2i}$ **20.** $\dfrac{\sqrt{2}+\sqrt{6}i}{-1+\sqrt{3}i}$

In Problems 21–26, use (8) to compute the indicated power.

21. $(1+\sqrt{3}i)^9$ **22.** $(2-2i)^5$

23. $(\frac{1}{2}+\frac{1}{2}i)^{10}$ **24.** $(-\sqrt{2}+\sqrt{6}i)^4$

25. $\left(\cos\dfrac{\pi}{8}+i\sin\dfrac{\pi}{8}\right)^{12}$

26. $\left[\sqrt{3}\left(\cos\dfrac{2\pi}{9}+i\sin\dfrac{2\pi}{9}\right)\right]^6$

In Problems 27–32, use (10) to compute all roots. Sketch these roots on an appropriate circle centered at the origin.

27. $(8)^{1/3}$ **28.** $(1)^{1/8}$

29. $(i)^{1/2}$ **30.** $(-1+i)^{1/3}$

31. $(-1+\sqrt{3}i)^{1/2}$ **32.** $(-1-\sqrt{3}i)^{1/4}$

In Problems 33 and 34, find all solutions of the given equation.

33. $z^4+1=0$ **34.** $z^8-2z^4+1=0$

In Problems 35 and 36, express the given complex number first in polar form and then in the form $a+ib$.

35. $\left(\cos\dfrac{\pi}{9}+i\sin\dfrac{\pi}{9}\right)^{12}\left[2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)\right]^5$

36. $\dfrac{\left[8\left(\cos\dfrac{3\pi}{8}+i\sin\dfrac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\dfrac{\pi}{16}+i\sin\dfrac{\pi}{16}\right)\right]^{10}}$

37. Use the result $(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta$ to find trigonometric identities for $\cos 2\theta$ and $\sin 2\theta$.

38. Use the result $(\cos\theta+i\sin\theta)^3=\cos 3\theta+i\sin 3\theta$ to find trigonometric identities for $\cos 3\theta$ and $\sin 3\theta$.

39. (a) If $z_1=-1$ and $z_2=5i$, verify that

$$\operatorname{Arg}(z_1z_2)\neq\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2).$$

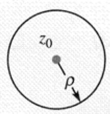
(b) If $z_1=-1$ and $z_2=-5i$, verify that

$$\operatorname{Arg}(z_1/z_2)\neq\operatorname{Arg}(z_1)-\operatorname{Arg}(z_2).$$

40. For the complex numbers given in Problem 39, verify in both parts **(a)** and **(b)** that

$$\arg(z_1z_2)=\arg(z_1)+\arg(z_2) \quad\text{and}\quad \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2).$$

17.3 Sets of Points in the Complex Plane



$|z-z_0|=\rho$

Figure 17.6

■ Terminology Before discussing the concept of functions of a complex variable, we need to introduce some essential terminology about sets in the complex plane.

Suppose $z_0=x_0+iy_0$. Since $|z-z_0|=\sqrt{(x-x_0)^2+(y-y_0)^2}$ is the distance between the points $z=x+iy$ and $z_0=x_0+iy_0$, the points $z=x+iy$ that satisfy the equation

$$|z-z_0|=\rho,$$

$\rho>0$, lie on a **circle** of radius ρ centered at the point z_0. See Figure 17.6.

Example 1 **Circles**

(a) $|z|=1$ is the equation of a unit circle centered at the origin.
(b) $|z-1-2i|=5$ is the equation of a circle of radius 5 centered at $1+2i$. ❑

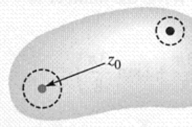


Figure 17.7

The points z satisfying the inequality $|z-z_0|<\rho$, $\rho>0$, lie within, but not on, a circle of radius ρ centered at the point z_0. This set is called a **neighborhood** of z_0 or an **open disk.** A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhoood of z_0 that lies entirely within S. If every point z of a set S is an interior point, then S is said to be an **open set.** See Figure 17.7. For example, the inequality $\operatorname{Re}(z)>1$ defines a *right half-plane,* which is an open set. All complex numbers $z=x+iy$ for which $x>1$ are in this set. If we choose, for example, $z_0=1.1+2i$, then a neighborhood of z_0 lying entirely in the set is defined by $|z-(1.1+2i)|<0.05$. See Figure 17.8. On the other hand, the set S of points in the complex plane defined by $\operatorname{Re}(z)\geq 1$ is *not* open, since every neighborhood of a point on the line $x=1$ must contain points in S and points not in S. See Figure 17.9.

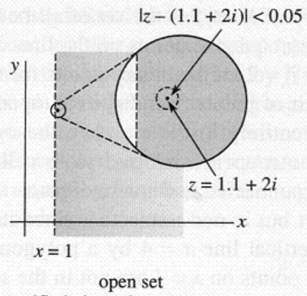


Figure 17.8

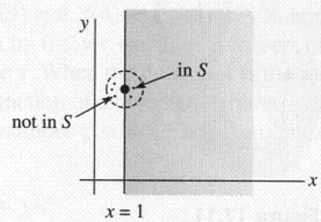


Figure 17.9

Example 2 **Open Sets**

Figure 17.10 illustrates some additional open sets.

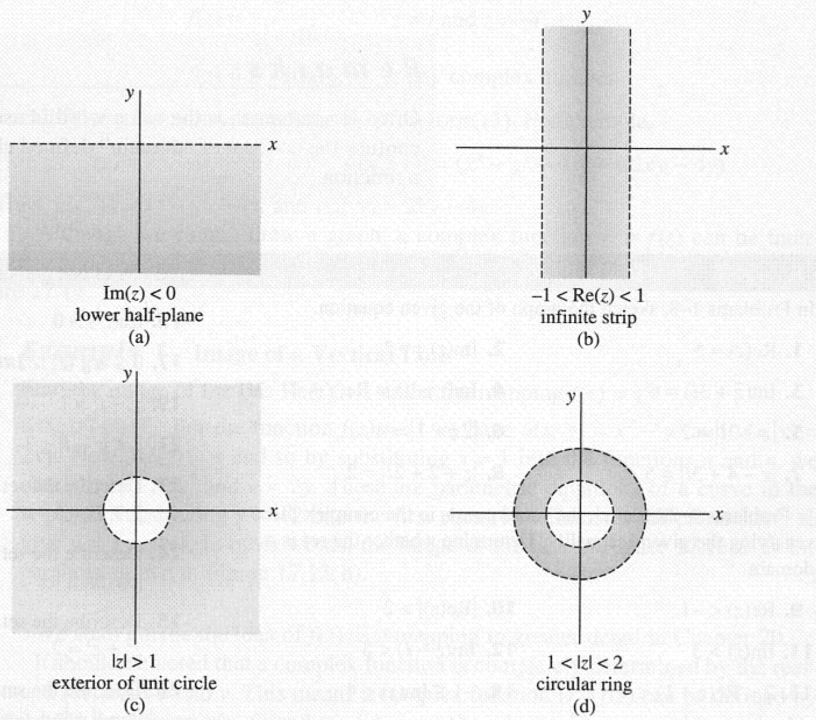


Figure 17.10 ❑

The set of numbers satisfying the inequality

$$\rho_1 < |z - z_0| < \rho_2,$$

such as illustrated in Figure 17.10(d), is also called an open **annulus.**

If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S, then z_0 is said to be a **boundary point** of S.

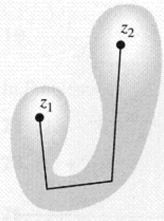


Figure 17.11

The **boundary** of a set S is the set of all boundary points of S. For the set of points defined by $\text{Re}(z) \geq 1$, the points on the line $x = 1$ are boundary points. The points on the circle $|z - i| = 2$ are boundary points for the disk $|z - i| \leq 2$.

If any pair of points z_1 and z_2 in an open set S can be connected by a polygonal line that lies entirely in the set, then the open set S is said to be **connected.** See Figure 17.11. An open connected set is called a **domain.** All the open sets in Figure 17.10 are connected and so are domains. The set of numbers satisfying $\text{Re}(z) \neq 4$ is an open set but is not connected, since it is not possible to join points on either side of the vertical line $x = 4$ by a polygonal line without leaving the set (bear in mind that the points on $x = 4$ are not in the set).

A **region** is a domain in the complex plane with all, some, or none of its boundary points. Since an open connected set does not contain any boundary points, it is automatically a region. A region containing all its boundary points is said to be **closed.** The disk defined by $|z - i| \leq 2$ is an example of a closed region and is referred to as a closed disk. A region may be neither open nor closed; the annular region defined by $1 \leq |z - 5| < 3$ contains only some of its boundary points and so is neither open nor closed.

Remarks

Often in mathematics the same word is used in entirely different contexts. Do not confuse the concept of "domain" defined above with the concept of the "domain of a function."

EXERCISES 17.3

Answers to odd-numbered problems begin on page A-71.

In Problems 1–8, sketch the graph of the given equation.

1. $\text{Re}(z) = 5$

2. $\text{Im}(z) = -2$

3. $\text{Im}(\bar{z} + 3i) = 6$

4. $\text{Im}(z - i) = \text{Re}(z + 4 - 3i)$

5. $|z - 3i| = 2$

6. $|2z + 1| = 4$

7. $|z - 4 + 3i| = 5$

8. $|z + 2 + 2i| = 2$

In Problems 9–22, sketch the set of points in the complex plane satisfying the given inequality. Determine whether the set is a domain.

9. $\text{Re}(z) < -1$

10. $|\text{Re}(z)| > 2$

11. $\text{Im}(z) > 3$

12. $\text{Im}(z - i) < 5$

13. $2 < \text{Re}(z - 1) < 4$

14. $-1 \leq \text{Im}(z) < 4$

15. $\text{Re}(z^2) > 0$

16. $\text{Im}(1/z) < \frac{1}{2}$

17. $0 \leq \arg(z) \leq 2\pi/3$

18. $|\arg(z)| < \pi/4$

19. $|z - i| > 1$

20. $|z - i| > 0$

21. $2 < |z - i| < 3$

22. $1 \leq |z - 1 - i| < 2$

23. Describe the set of points in the complex plane that satisfies $|z + 1| = |z - i|$.

24. Describe the set of points in the complex plane that satisfies $|\text{Re}(z)| \leq |z|$.

25. Describe the set of points in the complex plane that satisfies $z^2 + \bar{z}^2 = 2$.

26. Describe the set of points in the complex plane that satisfies $|z - i| + |z + i| = 1$.