4. Convergence of standard Fourier series in $L^2(S^1)$

We continue to work with the space of L^2 functions on the circle S^1 with circumference 1. When convenient, we'll identify this space with periodic functions on $\mathbb{R}$ with period 1. The main point of this section is to prove that the standard basis for Fourier series on S^1, namely

$$e_n = e^{2n\pi ix}, \qquad n \in \mathbb{Z},$$

is an orthonormal basis for $L^2(S^1)$. In the course of our proof, we'll need to consider the various spaces $C^k(S^1)$ of k-times differentiable functions on S^1 — keep in mind that if we consider these functions on a unit interval like $[0,1]$ or $[-\frac{1}{2}, \frac{1}{2}]$, then the values of the functions *and their derivatives up to order* k must agree at the endpoints of the interval, so that the function can be extended to a periodic function with period 1 that is C^k everywhere. We'll also occasionally use the space $C^\infty(S^1)$ of infinitely-differentiable functions. Recall that by Exercise 11 in section 2, all of the spaces $C^k(S^1)$ are dense in $L^2(S^1)$. The space $C^0(S^1)$ is simply the space of continuous functions on S^1 – it comes with its own notion of length, called the "sup norm" (or "uniform norm" or L^∞-norm):

$$\|f\|_\infty = \max_{0\le x\le 1} |f(x)|.$$

The subscript ∞ serves to distinguish this notion of length from the usual L^2-norm. Note we have:

$$\|f\|_2 = \left(\int_0^1 |f|^2\right)^{1/2} \le \|f\|_\infty.$$

Theorem 1: *The orthonormal family*

$$e_n = e^{2n\pi ix}, \qquad n \in \mathbb{Z},$$

is a basis for $L^2(S^1)$. *In other words, every function* $f \in L^2(S^1)$ *can be expanded into a Fourier series*

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$$

with coefficients

$$\hat{f}(n) = \langle f, e_n\rangle = \int_0^1 f\bar{e}_n = \int_0^1 f(x)e^{-2n\pi ix}dx,$$

where the sum is understood in the sense of $L^2(S^1)$. *By Theorem 3 of section 3, the map* $f \mapsto \hat{f}$ *is an isomorphism of* $L^2(S^1)$ *onto* $L^2(\mathbb{Z})$ *and there is a Plancherel identity:*

$$\|f\|_2^2 = \int_0^1 |f|^2 = \|\hat{f}\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Convention: From now on, we'll write e_n for $e^{2n\pi ix}$.

Exercise 1: Show that the family consisting of $f_0 = 1$, $f_n = \sqrt{2}\cos 2n\pi x$, $g_n = \sqrt{2}\sin 2n\pi x$, $n = 1, \ldots$ is also an orthonormal family in $L^2(S^1)$. Deduce from Theorem 1 that this is a basis and that the (perhaps complex) Fourier series for f can also be expressed in "real form"

$$f = \hat{f}_{even}(0) + \sum_{n=1}^{\infty} \hat{f}_{even}(n)\sqrt{2}\cos 2n\pi x + \hat{f}_{odd}(n)\sqrt{2}\sin 2n\pi x,$$

with coefficients

$$\begin{aligned}
\hat{f}_{even}(0) &= \int_0^1 f(x)\,dx \\
\hat{f}_{even}(n) &= \sqrt{2}\int_0^1 f(x)\cos 2n\pi x\,dx \\
\hat{f}_{odd}(n) &= \sqrt{2}\int_0^1 f(x)\sin 2n\pi x\,dx
\end{aligned}$$

for $n \geq 1$.

The main step in the proof that the exponentials e_n actually span $L^2(S^1)$ is to check that the Fourier series of a smooth function f actually converges to f. This is the content of the following theorem:

Theorem 2: *For any $k \geq 1$, and any $f \in C^k(S^1)$, the partial sums*

$$S_n = S_n(f) = \sum_{j=-n}^{n} \hat{f}(j)e_j$$

converge to f uniformly as $n \to \infty$. In fact, $\|S_n - f\|_\infty$ is bounded by a constant multiple of $n^{\frac{1}{2}-k}$.

Remark: The last sentence, with the bound on $\|S_n - f\|_\infty$, indicates that the speed of convergence of a Fourier series improves with the smoothness of f. This is a reflection of the fact that *local* properties of f (such as differentiability) are reflected in *global* properties of $\hat{f}$ (such as rapid decay as $n \to \pm\infty$). This "local-global duality" is a hallmark of the theory of Fourier series and integrals, as we will see over and over again.

Proof of Theorem 2: We'll do the proof here for $k = 1$ and leave the larger values of k as an exercise. To begin with, if $f \in C^1(S^1)$, we can write

$$\widehat{f'}(n) = \int_0^1 f'\overline{e}_n = -\int_0^1 f\overline{e}'_n = 2n\pi i\hat{f}(n),$$

using integration by parts. So by Schwartz' inequality and Bessel's inequality, if $m \le n < \infty$, then

$$\begin{aligned} |S_m(x) - S_n(x)| &\le \sum_{|j|>m} |\hat{f}(j)| = \sum_{|j|>m} \frac{|\widehat{f'}(j)|}{|2\pi j|} \\ &\le \left(\sum_{|j|\ge m} |\widehat{f'}(j)|^2 \right)^{1/2} \left(\sum_{|j|\ge m} \frac{1}{(2\pi j)^2} \right)^{1/2} \\ &\le \|f'\|_2 \times \text{a constant multiple of } n^{-1/2} \end{aligned}$$

This shows that the sequence S_n of partial sums converges uniformly, and at the advertised speed, to *something*. But *to what*?

To show that S_n in fact converges to f, we have to use the *Dirichlet kernel* function

$$\begin{aligned} D_n(x) = \sum_{j=-n}^{n} e_j(x) &= \sum_{j=-n}^{n} e^{2j\pi i x} \\ &= e^{-2n\pi i x} \sum_{j=0}^{2n} e^{2j\pi i x} \\ &= e^{-2n\pi i x} \frac{e^{2\pi i(2n+1)x} - 1}{e^{2\pi i x} - 1} \\ &= \frac{\sin(2n+1)\pi x}{\sin \pi x}, \end{aligned}$$

with the understanding that $D_n(0) = 2n + 1$. Note that we automatically obtain the value of the intimidating-looking integral

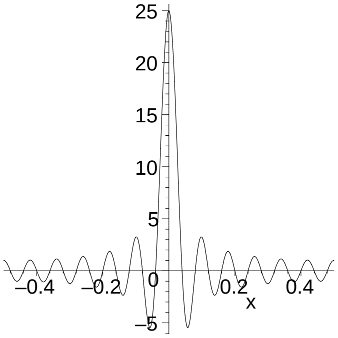
$$\int_0^1 D_n = \int_0^1 \frac{\sin(2n+1)\pi x}{\sin \pi x}\, dx = \sum_{j=-n}^{n} \int_0^1 e_j = 1.$$

Using D_n, we can express S_n in a simpler and more useful way:

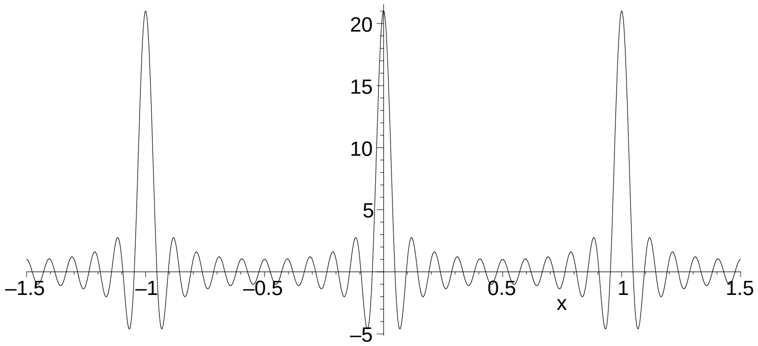
$$\begin{aligned} S_n(x) = \sum_{j=-n}^{n} \hat{f}(j) e_j(x) &= \sum_{j=-n}^{n} e_j(x) \int_0^1 f(y) \overline{e}_j(y)\, dy \\ &= \int_0^1 \sum_{j=-n}^{n} e_j(x-y) f(y)\, dy \\ &= \int_0^1 D_n(x-y) f(y)\, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+y) D_n(y)\, dy \end{aligned} .$$

The last line comes from the substitution $u = y - x$, the fact that D_n is an even function, and the fact that D_n and f are periodic with period 1, so we can choose any interval of length 1 to integrate over (Exercise: Fill in the details of this).

To see how we are going to verify that S_n is a good approximation of f for large n is easier to understand with a picture of D_n in mind. For instance, here is a graph of D_{12}:



To see the periodicity better, here is a picture of D_{10} over several periods:



As n gets larger, the peak of D_n tends to ∞, while the oscillations become increasingly rapid. While they do not die away, you might hope that their positive and negative parts will on average cancel each other out, with the result that the major contribution to the integral comes from a small neighborhood of $y = 0$. Our proof will take advantage (somewhat indirectly) of this phenomenon.

To begin, because

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} D_n = 1,$$

the difference between f and the partial sum S_n can be expressed as

$$\begin{aligned} S_n(x) - f(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+y) - f(x)] D_n(y)\, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} H(x,y) \sin(2n+1)\pi y\, dy, \end{aligned}$$

where

$$H(x,y) = \frac{f(x+y) - f(x)}{\sin \pi y}$$

for $y \neq 0$, and $H(x,0) = f'(x)/\pi$. Now fix a particular x in the interval $[-\frac{1}{2}, \frac{1}{2})$, and consider $H(x,y)$ as a function of y. As such, $H(x,y)$ belongs to $L^2([-\frac{1}{2}, \frac{1}{2})$, and

$$\begin{aligned} S_n(x) - f(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} H(x,y) \frac{e^{2n\pi iy} e^{\pi iy} - e^{-2n\pi iy} e^{-\pi iy}}{2i}\, dy \\ &= \frac{1}{2i}(\widehat{H_1}(-n) - \widehat{H_2}(n)) \end{aligned}$$

where $H_1(x,y) = e^{\pi iy} H(x,y)$ and $H_2(x,y) = e^{-\pi iy} H(x,y)$. But now we can use Bessel's inequality to assert

$$\sum_{n=-\infty}^{\infty} |\widehat{H_\nu}(n)|^2 \leq \|H\|_2^2 < \infty$$

for $\nu = 1, 2$. But because the nth term of a convergent series must go to zero, this shows that $H_\nu(n) \to 0$ as $n \to \pm\infty$. And this proves that

$$\lim_{n \to \infty} S_n(x) = f(x)$$

for each fixed x. So $S_n \to f$ pointwise. Now we can take $m \to \infty$ in the preceding estimate for $|S_n - S_m|$, which shows that $\|S_n - f\|_\infty$ is bounded by a constant multiple of $n^{-1/2}$. This completes the proof of Theorem 2 for $k = 1$.

Exercise 2: Finish the proof of Theorem 1, using the fact that $C^\infty(S^1)$ is dense in $L^2(S^1)$.

Exercise 3: Finish the proof of Theorem 2 for $2 \leq k < \infty$. (Hint: Use induction, the fact that $\widehat{f'}(n) = 2\pi i \hat{f}(n)$, and the fact that $\widehat{f'}(0) = 0$ – but you should prove this last fact).

Exercise 4: Show that $f \in C^\infty(S^1)$ if and only if $\hat{f}$ is *rapidly decreasing*, in the sense that $n^p \hat{f}(n) \to 0$ as $n \to \pm\infty$ for every finite p. Hint: For one direction, use that if $\hat{f}$ is rapidly decreasing, then $\sum \hat{f}(n) e_n'$ converges uniformly to a periodic function f_1, and

$$\int_0^x f_1 = \sum \hat{f}(n) \int_0^x e_n' = \sum \hat{f}(n)[e_n(x) - e_n(0)] = f(x) - f(0).$$

The question of pointwise convergence of the Fourier series of a function in $L^2(S^1)$ has a long and interesting history. It can be very complicated in general. In 1966, Carleson proved that it must converge a.e., but there are examples of $f \in C(S^1)$ for which the sum diverges at uncountably many points. But there is another kind of convergence, called *Cesaro summability*, that we can count on:

Theorem 3: *If the function f is in $C(S^1)$, then the arithmetic means $(S_0 + S_1 + \cdots + S_{n-1})/n$ of the partial sums $S_n = \sum_{|k| \leq n} \hat{f}(k) e_k$ converge uniformly to f.*

Exercise 5: Show that if the numerical sequence $c_0, c_1, \ldots$ converges to y, then the arithmetic means

$$\frac{c_0 + c_1 + \cdots c_{n-1}}{n}$$

also converge to y. Give an example to show that the latter limit can exist even if $\lim_{n\to\infty} c_n$ does not.

Proof of Theorem 3: The main ingredient in the proof is the arithmetic mean of the Dirichlet kernel:

$$\begin{aligned} F_n &= \frac{1}{n}(D_0 + \cdots + D_{n-1}) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \frac{\sin(2k+1)\pi x}{\sin \pi x} \\ &= \frac{1}{n}\left(\frac{\sin n\pi x}{\sin \pi x}\right)^2 . \end{aligned}$$

Exercise 6: Check the summation (Hint: Think of $\sin(2k+1)\pi x$ as the imaginary part of a complex exponential and then sum the resulting geometric series.)

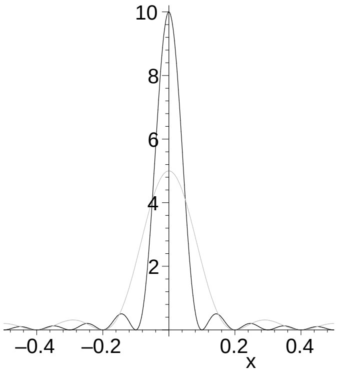
F_n is called the *Fejér kernel.* Note that F_n is a non-negative function and that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n = \frac{1}{n}\sum_{k=0}^{n-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} D_k = 1.$$

Now we have to check that the difference

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} S_k(x) - f(x) &= \frac{1}{n}\sum_{k=0}^{n-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+y)D_k(y)dy - f(x) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+y)F_n(y)\,dy - f(x) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+y) - f(x)]F_n(y)\,dy \end{aligned}$$

is uniformly small. This is easier for the Fejér kernel F_n than it was for the Dirichlet kernel D_n, because the tails of F_n are small for large n, and not just negligible due to rapid oscillation. For instance, here are graphs of F_5 and F_{10}:



To show that the difference between the nth Cesaro sum and f is uniformly small, we'll first note that because f is continuous on the circle (a compact set), f is *uniformly* continuous, so by choosing y small enough we can make $\|f_y - f\|_\infty$ as small as we like, where we define $f_y(x) = f(x+y)$ (it will be convenient to have this definition around once in a while in the future – watch for it). So let's suppose $\varepsilon > 0$ is given, and we'll choose δ so that $\|f_y - f\| < \varepsilon/2$ for all y with $|y| < \delta$.

Now we'll break the integral above into two parts, one for $|y| < \delta$ and the other for $|y| \geq \delta$. For the first part, we have:

$$\begin{aligned}\left|\int_{-\delta}^{\delta}[f(x+y)-f(x)]F_n(y)\,dy\right| &= \left|\int_{-\delta}^{\delta}[f_y(x)-f(x)]F_n(y)\,dy\right| \\ &\leq \int_{-\delta}^{\delta}|f_y(x)-f(x)|F_n(y)\,dy \\ &\leq \|f_y - f\|_\infty \int_{-\delta}^{\delta} F_n(y)\,dy \\ &\leq \|f_y - f\|_\infty < \varepsilon/2.\end{aligned}$$

On the other part, we have:

$$\begin{aligned}\left|\int_{\frac{1}{2}\geq|y|\geq\delta}[f(x+y)-f(x)]F_n(y)\,dy\right| &= \left|\int_{\frac{1}{2}\geq|y|\geq\delta}[f_y(x)-f(x)]F_n(y)\,dy\right| \\ &\leq \int_{\frac{1}{2}\geq|y|\geq\delta}|f_y(x)-f(x)|F_n(y)\,dy \\ &\leq 2\|f_y - f\|_\infty \int_{\delta}^{\frac{1}{2}} F_n(y)\,dy \\ &\leq 4\|f\|_\infty \int_{\delta}^{\frac{1}{2}} \frac{1}{n}\left(\frac{\sin n\pi x}{\sin \pi x}\right)^2\,dy \\ &\leq \frac{2}{n\sin^2 \pi\delta}\|f\|_\infty,\end{aligned}$$

which we can force to be less than $\varepsilon/2$ by choosing n sufficiently large, because a specific $\delta > 0$ was chosen first. This completes the proof

Exercise 7: Show that for any $f \in L^2(S^1)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right) = \hat{f}(0) = \int_0^1 f$$

in the L^2 sense. (Hint: Compute the Fourier coefficients of the sum on the left, and then use the Plancharel identity).

Exercise 8: Show that for any $f \in C(S^1)$ and $0 \le r < 1$,

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e_n(\theta) = \int_0^1 \frac{1-r^2}{1-2r\cos 2\pi(\theta-\varphi)+r^2} f(\varphi)\, d\varphi.$$

(Remember this? Hint: Express $\hat{f}(n)$ as an integral and exchange the sum and the integral.)

Exercise 9: Prove that for any $f \in C(S^1)$,

$$\lim_{r\to 1^-} \|f - \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e_n\|_\infty = 0.$$

This is the Poisson formula for the solution of Dirichlet's problem for the Laplace equation on the disk. This proves that the solution actually takes on the prescribed boundary values. Prove and use the fact that:

$$\int_0^1 \frac{1-r^2}{1-2r\cos 2\pi\varphi + r^2}\, d\varphi = \sum_{n=-\infty}^{\infty} \hat{1}(n) r^{|n|} e_n(0) = 1$$

(here, $\hat{1}(n)$ is the nth Fourier coefficient of the constant function 1.)

For the next topic, we're going to consider yet another space of functions, the space $L^1(S^1)$ of (complex-valued) summable functions on the circle. In general, the space of L^1 functions on any set is the set of all functions f for which

$$\|f\|_1 = \int |f|$$

is finite. This new kind of "length" measurement for functions allows you to define a distance between functions (it is not the same as the L^2 distance), and it is easy to see that the trangle inequality holds:

$$\|f+g\|_1 = \int |f+g| \le \int |f| + \int |g| = \|f\|_1 + \|g\|_1,$$

so that L^1 is closed under addition of functions, and of course also under scalar multiplication by constants. Therefore, L^1 is a linear space endowed with a distance function (or norm).

Of course, as before, the elements of L^1 are not actually functions, but rather are equivalance classes of functions, where two functions are identified if the set on which they differ has measure zero. But we won't ever have to worry about this.

Exercise 10: Show that $L^1(S^1)$ is *not* a Hilbert space (because the norm does not come from an inner product). Hint: Show that in a Hilbert space we must have $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$, and then see if this works for the first two functions you can think of on the interval $[0,1]$.

Exercise 11: Show that $L^1(S^1)$ is complete and separable. (So it is what is called a *Banach space.*) This is like our earlier proof for L^2.

Exercise 12: Show that $C^\infty(S^1)$ is dense in $L^1(S^1)$. Also, show that $L^2(S^1) \subset L^1(S^1)$ (use the Schwarz inequality by writing f as $f \cdot 1$ to show that $\|f\|_1^2 \leq \|f\|_2^2$).

Exercise 13: Show that the inclusion in Exercise 12 is *proper*, in other words, find an L^1 function that is not L^2.

Exercise 14: Give examples to show that neither $L^2(\mathbb{R})$ nor $L^1(\mathbb{R})$ is contained in the other. (The issue is that $\mathbb{R}$ is unbounded.)

Exercise 15: Prove the Riesz representation theorem for $L^1(S^1)$. That is, show that any linear map L from $L^1(S^1)$ into the complex numbers that satisfies $|L(f)| \leq C\|f\|_1$ for a constant C independent of f, can be expressed as $L(f) = \int f\overline{g}$, wher g is a bounded measurable function. (Hint: recall that $L^2(S^1) \subset L^1(S^1)$, and use the L^2 Riesz representation theorem to find $g \in L^2$. Then check that $\int_A |g| \leq C\mu(A)$ for any measurable $A \subset S^1$. Also, L^2 is *dense* in L^1).

Now, for any $f \in L^1(S^1)$, the function $f\overline{e}_n$ is summable (because $|e_n(x)| = 1$ for all x), and so we can calculate Fourier coefficients for f and make a Fourier series for f as usual:

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$$

with coefficients

$$\hat{f}(n) = \int_0^1 f\overline{e}_n = \int_0^1 f(x) e^{-2n\pi i x}\, dx.$$

(We don't write $\langle f, e_n \rangle$ because the inner product doesn't apply in L^1.) But in what sense will these series converge? The answer is given in the following theorem.

Theorem 4: *The Cesaro sums of the Fourier series of f, i.e., the arithmetic means of the partial sums $S_N = \sum_{|n|\leq N} \hat{f}(n) e_n$, namely*

$$\frac{1}{n}(S_0 + \cdots + S_{n-1}),$$

converge to f in the sense of distance in $L^1(S^1)$:

$$\lim_{N\to\infty} \|\frac{1}{N}(S_0 + \cdots + S_{n-1})\|_1 = 0.$$

In particular, the map $f \mapsto \hat{f}$ is one-to-one.

Proof: The model here is the proof of Theorem 3. Just as we did there, we can express the difference between f and the Nth Cesaro sum in terms of the Fejér kernel:

$$\frac{1}{N}(S_0 + \cdots + S_{N-1}) - f = \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+y) - f(x)] F_N(y)\, dy,$$

and so the L^1-size of the difference can be estimated as follows:

$$\begin{aligned} \|\frac{1}{N}(S_0 + \cdots + S_{N-1}) - f\|_1 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+y) - f(x)] F_N(y)\, dy \right| dx \\ &\le \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x+y) - f(x)| F_N(y)\, dy\, dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x+y) - f(x)|\, dx \right) F_N(y)\, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \|f_y - f\|_1 F_N(y)\, dy, \end{aligned}$$

where $f_y(x) = f(x+y)$ as before. The rest of the proof runs parallel to Theorem 3. The only new ingredient is the following:

Exercise 16: Show that for $f \in L^1(S^1)$, the map $y \mapsto f_y$ is continuous in the sense that

$$\lim_{y\to 0} \|f_y - f\|_1 = 0.$$

(Hint: $\|f_y - f\|_1 \le \|f_y - f\|_\infty$ if f is continuous, and the continous functions are dense in $L^1(S^1)$.)

The Fourier conefficients of an L^1 functions do not necessarily satisfy $\sum |\hat{f}(n)|^2 < \infty$, because this happens only if $f \in L^2$. But the Fourier coefficients of an L^1 function are certainly bounded:

$$|\hat{f}(n)| = \left| \int_0^1 f \overline{e}_n \right| \le \int_0^1 |f| = \|f\|_1.$$

This can be improved, as we have seen before. The *Riemann-Lebesgue lemma* is true for $f \in L^1$:

Theorem 5: *The Fourier coeffieicnts $\hat{f}(n)$ of the function $f \in L^1(S^1)$ tend to 0 as $|n| \to \infty$.*

Proof: We can rewrite $\hat{f}(n)$ as

$$\begin{aligned}\hat{f}(n) = \int_0^1 f(x)e^{-2n\pi i}\,dx &= -\int_0^1 f(x)e^{-2n\pi ix}e^{-\pi ix}\,dx \\ &= -\int_0^1 f(x)e^{-2n\pi i(x-\frac{1}{2n})}\,dx \\ &= -\int_0^1 f(x+\frac{1}{2n})e^{-2n\pi ix}\,dx.\end{aligned}$$

Now average the two expressions for $\hat{f}(n)$ and estimate as follows:

$$\begin{aligned}|\hat{f}(n)| &= \frac{1}{2}\left|\int_0^1 (f(x)-f(x+\frac{1}{2n}))e^{-2n\pi ix}\,dx\right| \\ &\le \frac{1}{2}\int_0^1 |f(x)-f(x+\frac{1}{2n})|\,dx \\ &= \frac{1}{2}\|f-f_{\frac{1}{2n}}\|_1.\end{aligned}$$

By Exercise 16, this approaches 0 as $|n| \to \infty$. This completes the proof.

The expressions we derived for the partial sums of a Fourier series and for the Cesaro means of the partial sums of a Fourier series, namely

$$\int_0^1 D_n(x-y)f(y)\,dy \quad \text{and} \quad \int_0^1 F_n(x-y)f(y)\,dy$$

are examples of a new kind of "product" of functions, called the *convolution product*, that we will study next. The space $L^1(S^1)$ is an algebra under this multiplication, defined as follows: For two functions f and g, we define a new function

$$f * g(x) = \int_0^1 f(x-y)g(y)\,dy.$$

Now, for a particular value of x , it may be that the integral does not exist, so we have to check that $f * g$ makes sense as an L^1 function. To justify this, we first look at the double integral:

$$I = \int_0^1\int_0^1 |f(x-y)g(y)|dx\,dy.$$

The integrand $|f(x-y)g(y)|$ is a non-negative function that is integrable, and $I \le \infty$, and for such integrals, Fubini's theorem says you can evaluate them as iterated integrals:

$$I = \int_0^1 |g(y)|\int_0^1 |f(x-y)|\,dx\,dy = \int_0^1 |g|\int_0^1 |f| = \|f\|_1\|g\|_1 < \infty.$$

Also, since

$$I = \int_0^1\int_0^1 |f(x-y)g(y)|dy\,dx < \infty,$$

we can use Fubini to conclude that $f(x-y)g(y)$ is summable as a function of y for almost every $x \in [0,1)$. Thus, $f * g(x)$ is defined by an "honest" Lebesgue integral for almost every x and is itself a periodic summable function:

$$\|f * g\|_1 \le I = \|f\|_1 \|g\|_1 < \infty.$$

Exercise 17: Show that the product $f * g$ is associative and commutative (you could do this for continuous functions, and then use the fact that the continuous functions are dense in L^1).

Exercise 18: Show that $L^2(S^1)$ is an *ideal* in $L^1(S^1)$ – this means that if *either* f or g belongs to L^2, then $f * g$ belongs to L^2 as well.

The main reason for introducing the convolution product is its amazing relationship to Fourier series:

$$\widehat{f * g} = \hat{f}\hat{g},$$

which plays a very important role in applications of Fourier series.

Proof: We'll use Fubini's theorem to do this:

$$\begin{aligned}
\widehat{f * g}(n) &= \int_0^1 \left(\int_0^1 f(x-y) g(y)\, dy \right) \overline{e}_n(x)\, dx \\
&= \int_0^1 \int_0^1 f(x-y)\overline{e}_n(x-y) g(y) \overline{e}_n(y)\, dx\, dy \\
&= \int_0^1 g(y)\overline{e}_n(y) \left(\int_0^1 f(x-y)\overline{e}_n(x-y)\, dx \right) dy \\
&= \int_0^1 g\overline{e}_n \int_0^1 f\overline{e}_n \\
&= \hat{f}(n)\hat{g}(n).
\end{aligned}$$

Exercise 19: Explain why $L^1(S^1)$ does not have a (convolution) multiplicative identity. (Hint: A multiplicative identity e would satisfy $e * f = f$ Now think about what $\hat{e}$ would have to be, keeping the Riemann-Lebesgue lemma in mind.)

Exercise 20: Let f^{*n} be the n-fold product $f * \cdots * f$ of an L^2 function f. Prove that

$$\lim_{n\to\infty} (\|f^{*n}\|_1)^{1/n} = \max_n |\hat{f}(n)|.$$

(This is actually true for L^1 functions, but the proof is harder.) Hint: It's easy to show that the right side is less than or equal to the left. To go the other way, use the fact that, if $g = \exp(-i \arg f^{*n})$, then

$$\|f^{*n}\|_1 = \int f^{*n} g = \sum \widehat{f^{*n}} \hat{g}.$$