Two things
Monday, February 2

Once and for all:

Proposition. Suppose $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at every point of a convex set $E \subset \mathbf{R}^n$. If $\mathbf{a}$ and $\mathbf{b}$ are two points of E, then

$$|f(\mathbf{a}) - f(\mathbf{b})| \leq M|\mathbf{a} - \mathbf{b}|$$

where $M = \sup_{\mathbf{x}\in E} \|Df(\mathbf{x})\|$.

Proof. As we did in class, define

$$\mathbf{x}(t) = t\mathbf{a} + (1-t)\mathbf{b},$$

so that $D\mathbf{x} = \mathbf{a} - \mathbf{b}$ and the derivative of the composition $\varphi(t) = f(\mathbf{x}(t))$ (which maps $\mathbf{R}$ to $\mathbf{R}^m$) with respect to t is

$$D\varphi(t) = Df(\mathbf{x}(t))(\mathbf{a} - \mathbf{b}).$$

By the fundamental theorem of calculus applied to functions of one variable (since we haven't discussed multivariable integration yet),

$$f(\mathbf{a}) - f(\mathbf{b}) = \varphi(1) - \varphi(0) = \int_0^1 D\varphi(t)\, dt.$$

But

$$\left|\int_0^1 D\varphi(t)\, dt\right| \leq \int_0^1 |D\varphi(t)|\, dt = \int_0^1 |Df(\mathbf{x}(t))(\mathbf{a}-\mathbf{b})|\, dt \leq \int_0^1 \|Df(\mathbf{x}(t))\|\, |\mathbf{a}-\mathbf{b}|\, dt.$$

The constant $|\mathbf{a}-\mathbf{b}|$ can be pulled out of the integral, and the integral of $\|Df(\mathbf{x}(t))\|$ is certainly less than M. This proves the proposition.

And another thing:

Proposition. The function

$$f(x,y) = (1-y)^3x^2 + y^2$$

has only one critical point, at $(x,y) = (0,0)$. This critical point is a strict local minimum. But it is not the global minimum – indeed, $f(1,11) = -900$.

You should verify the claim that the origin is the only critical point and that it is a strict local minimum (on Thursday, I had $(1-y^3)$ instead of $(1-y)^3$. Describe the set of points (x,y) for which $f(x,y) < f(0,0)$.