Math 509 **Dr. DeTurck**
Midterm 1 **Thursday, February 26**

1. Let $f(x)$ be a differentiable function that satisfies
$$f(x+y) = f(x)f(y)$$
for all real x and y. Prove that $f(x) = c^x$ where $c = f(1)$. (This is essentially problem 6(a) on page 197 of Rudin.)

2. (An easy one if you approach it the right way.) Let A be a rectangle (box) in $\mathbf{R}^n$, and let f and g be bounded, integrable functions on A. Show that the product fg is also integrable.

3. A function $f\colon \mathbf{R}^n \to \mathbf{R}$ is called *homogeneous of degree* m if for $t \in \mathbf{R}$ with $t \geq 0$ and $\mathbf{x} \in \mathbf{R}^n$, we have
$$f(t\mathbf{x}) = t^m f(\mathbf{x}).$$
(a) Give examples of non-zero functions that are homogenous of degree m for $m = 1, 2, 3$.

   (b) Show that if f is differentiable, as well as being homogeneous of degree m, then
$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(\mathbf{x}) = mf(\mathbf{x})$$
for all $\mathbf{x} \in \mathbf{R}^n$.

4. Let $f\colon \mathbf{R} \to \mathbf{R}$ and $g\colon \mathbf{R} \to \mathbf{R}$ be differentiable functions for which $f(1) = g(1) = 0$. For what additional conditions on f and g will the implicit function theorem guarantee a solution of the eqations
$$f(xy) + g(yz) = 0, \qquad g(xy) + f(yz) = 0$$
for y and z as functions of x in a neighborhood of the point $(1, 1, 1)$?

5. Are the functions $f(x, y) = (x + y)/x$ and $g(x, y) = (x + y)/y$ functionally dependent? Use the implicit function theorem to decide. If they are, then find a nontrivial function F so that $F(f, g) = 0$.

6. Consider the integral

$$I(a) = \int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$$

as a function of the parameter a, for $a > 0$.

   (a) Explain why (i.e., prove that – and cite carefully any theorem you use) $\lim_{a\to\infty} I(a) = 0$.

   (b) Explain why $I'(a)$ can be computed by differentiation under the integral sign.

   (c) Use part (b) to calculate $I'(a)$ — sorry about the integration by parts!

   (d) Now use (a) and (c) to calculate $I(a)$ for all $a > 0$.

   (e) *Extra credit*: Justify the interchange of the (improper!) integral with the process of taking the limit as $a \to 0$ and hence evaluate

   $$\int_0^\infty \frac{\sin x}{x}\,dx.$$

   If you took Math 241 or 410, you may know a complex variables way to do this integral as well.

7. Consider the equation

$$f(x,\varepsilon) = x^3 - 3x^2 + \varepsilon = 0.$$

   (a) Show that when $\varepsilon = 0$, the equation $f(x,0) = 0$ has two roots, and find them.

   (b) For ε near zero, near which of the two roots from part (a) is the existence of a solution of $f(x,\varepsilon) = 0$ guaranteed by the implicit function theorem?

   (c) Calculate $\dfrac{dx}{d\varepsilon}$ at $\varepsilon = 0$ and $x =$ the root from part (b).

   (d) The implicit function theorem only tells you that the solution $x(\varepsilon)$ exists only for ε near zero. Now, let's show that the function $x(\varepsilon)$ can be extended continuously (even smoothly) all the way to $\varepsilon = 1$. Do this as follows:

      (*i*) Prove the following using the implicit function theorem: If (x_0, ε_0) is a point in $\mathbf{R}^2$ for which $f(x_0,\varepsilon_0) = 0$, $0 \le \varepsilon \le 1$, and $2.5 \le x_0 \le 4$, then for all ε near ε_0, there is an x near x_0 such that $f(x,\varepsilon) = 0$, and in fact x is a continuous (smooth) function of ε near ε_0.

      (*ii*) Let I be the subinterval of the ε interval $[0,1]$ on which the solution $x(\varepsilon)$ from part (b) exists as a continuous function. Explain why part (d,*i*) implies that I is open.

(*iii*) Using any method (think "Math 104"), prove that for all ε between 0 and 1, there is exactly one solution x of $f(x, \varepsilon) = 0$ such that $2.5 \leq x \leq 4$.

(*iv*) Now suppose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ is an convergent (to ε^*) sequence of values of ε between 0 and 1 such that for each ε_i there is an x_i between 2.5 and 4 such that $f(\varepsilon_i, x_i) = 0$. Explain why there must be at least a subsequence of the x_i that converges to a number x^* that satisfies $2.5 \leq x^* \leq 4$.

(*v*) Explain why it must be the case that $f(x^*, \varepsilon^*) = 0$.

(*vi*) Explain why parts (d,*iv*) and (d,*v*) imply that the interval I from part (d,*ii*) is closed.

(*vii*) Explain why parts (d,*ii*) and (d,*vi*) together imply that the interval I is all of $[0, 1]$. Therefore there is a continuous path $(x(\varepsilon), \varepsilon)$ from the solution you found in part (b) for $\varepsilon = 0$ to the (unknown, but existent) solution for $\varepsilon = 1$.

This is called the *continuity method* for proving existence of solutions to nonlinear equations, and is often used in the study of ordinary and partial differential equations.