

1. First note that since  $f(0) = f(0 + 0) = f(0)f(0)$ , we have either  $f(0) = 0$  (in which case  $f(x) = f(x + 0) = f(x)f(0) = 0$  for all  $x$ , so  $c = 0$ ) or else  $f(0) = 1$ .

In the latter case, we can take the derivative of both sides of the defining equation with respect to  $y$ , holding  $x$  fixed, to obtain

$$f'(x + y) = f(x)f'(y).$$

For  $y = 0$  we thus obtain  $f'(x) = f'(0)f(x)$ . Since  $f'(0)$  is a constant, let's call it  $\ln c$ , we see that  $f$  satisfies the initial-value problem  $f' = (\ln c)f$ ,  $f(0) = 1$ , the unique solution of which is  $f(x) = e^{(\ln c)x} = c^x$ .

2. Since  $f$  and  $g$  are integrable, the sets  $S_f$  and  $S_g$  consisting of points in  $A$  where  $f$  and  $g$  are discontinuous have measure zero. But the subset  $S_{fg}$  of  $A$  consisting of points where the product  $fg$  is discontinuous is contained in the union of  $S_f$  and  $S_g$  (because the product of continuous functions is continuous), and so  $S_{fg}$  also has measure zero, thus  $fg$  is integrable.
3. (a) Some simple ones are  $x + y$ ,  $x^2 + xy$ ,  $xyz + z^3$ , but homogeneous functions need not be polynomials:  $\sqrt{x^4 + y^4}$  is homogeneous of degree 2, for instance.  
(b) Take the derivative of the equation

$$f(tx_1, \dots, tx_n) = t^m f(x_1, \dots, x_n)$$

with respect to  $t$  (holding  $x_1, \dots, x_n$  fixed), and get:

$$x_1 \frac{\partial f}{\partial x_1}(tx_1, \dots, tx_n) + \dots + x_n \frac{\partial f}{\partial x_n}(tx_1, \dots, tx_n) = mt^{m-1} f(x_1, \dots, x_n).$$

Now set  $t = 1$  to obtain the desired equation.

4. Let  $F(x, y, z) = f(xy) + g(yz)$  and  $G(x, y, z) = g(xy) + f(yz)$ . The implicit function theorem guarantees a solution for  $y$  and  $z$  in terms of  $x$  in a neighborhood of any point where the determinant of

$$\begin{bmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{bmatrix}$$

is not zero. Now

$$\frac{\partial F}{\partial y} = x f'(xy) + z g'(yz), \quad \frac{\partial F}{\partial z} = y g'(yz)$$

and

$$\frac{\partial G}{\partial y} = x g'(xy) + z f'(yz), \quad \frac{\partial G}{\partial z} = y f'(yz).$$

So at  $(x, y, z) = (1, 1, 1)$  we're looking at the determinant of

$$\begin{bmatrix} f'(1) + g'(1) & g'(1) \\ f'(1) + g'(1) & f'(1) \end{bmatrix},$$

which is  $(f'(1))^2 - (g'(1))^2$ . This is non-zero if and only if  $f'(1) \neq \pm g'(1)$ , which is the required additional condition.

5. By the implicit function theorem, the functions are functionally dependent if the rank of the matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

is zero. Since this is

$$\begin{bmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{bmatrix}$$

which is zero, the functions *are* functionally dependent. Indeed, since we can write  $f(x, y) = 1 + y/x$  and  $g(x, y) = 1 + x/y$ , we get  $g = 1 + 1/(f - 1)$ , which simplifies to  $fg - f - g = 0$ .

6. (a) Since  $|\sin x/x| \leq 1$  for all  $x$ , we have that  $|e^{-ax} \sin x/x| \leq e^{-ax}$  for all  $x \geq 0$ . Therefore

$$|I(a)| \leq \int_0^\infty e^{-ax} dx = 1/a,$$

which approaches zero as  $a \rightarrow \infty$ , so  $I(a) \rightarrow 0$  as well by the “squeeze theorem”.

- (b) This one can't be done simply by appealing to Theorem 9.42 in the book, because of the infinite interval of integration (after all, if you integrate  $\varepsilon$  from 0 to  $\infty$ , you still get  $\infty$ ). The key is to get inside the proof of Theorem 9.42, and produce an estimate for the difference between the derivative of the integrand and the difference quotient for the integrand that can be integrated from 0 to  $\infty$  and still give a small ( $\varepsilon$ -sized) result. Later on, we'll have something called the Lebesgue dominated-convergence theorem to help with this. Here we go:

Let  $g(a, x) = -e^{-ax} \sin x$  (this is the derivative of the integrand with respect to  $a$ ) and let

$$\psi(a, b, x) = \frac{e^{-ax} \sin x}{x} - \frac{e^{-bx} \sin x}{x}$$

(this is the difference quotient for the integrand). By the mean-value theorem,

$$\psi(a, b, x) = g(c, x)$$

for some  $c$  between  $a$  and  $b$ .

We have to show that

$$\lim_{b \rightarrow a} \int_0^\infty \psi(a, b, x) dx = \int_0^\infty g(a, x) dx$$

for all  $a > 0$ . So, given  $a > 0$  and  $\varepsilon > 0$ , we must find  $\delta > 0$  such that

$$\left| \int_0^\infty \psi(a, b, x) dx - \int_0^\infty g(a, x) dx \right| < \varepsilon$$

provided  $|a - b| < \delta$ .

First off, we'll insist that  $\delta < a/4$ , to guarantee that  $b > 0$  so that the integral of  $\psi$  is certain to converge (and more than that, see below).

From our mean-value reasoning above,

$$|g(a, x) - \psi(a, b, x)| = |g(a, x) - g(c, x)| \leq |g(a, x) - g(b, x)|$$

and, because  $|\sin x/x| \leq 1$ ,

$$|g(a, x) - g(b, x)| \leq |e^{-ax} - e^{-bx}| \leq e^{-ax/2} |e^{-ax/2} - e^{-bx+ax/2}|$$

and we know that  $bx - \frac{ax}{2} > \frac{ax}{4}$  because  $b > 3a/4$ .

Now we need a little calculus lemma: if  $x \geq 0$ ,  $p > 0$  and  $q > 0$ , then

$$|e^{-px} - e^{-qx}| \leq \frac{|p - q|}{re}$$

for some  $r$  between  $p$  and  $q$ . To prove this, use the mean-value theorem to conclude that  $|e^{-px} - e^{-qx}| = |p - q|xe^{-rx}$ , and then do the Calc I problem that shows that the maximum of  $xe^{-rx}$  for  $x \geq 0$  is  $1/(re)$ .

Using this lemma, we get that

$$|e^{-ax/2} - e^{-bx+ax/2}| \leq \frac{|b - a|}{re}$$

for some  $r$  between  $\frac{ax}{2}$  and  $bx - \frac{ax}{2}$ . So  $r$  is definitely bigger than  $a/4$ .

Putting the entire chain of inequalities together yields

$$|g(a, x) - \psi(a, b, x)| \leq e^{-ax/2} \frac{4|b - a|}{a}$$

provided  $|b - a| \leq a/4$ . Therefore,

$$\left| \int_0^\infty g(a, x) - \psi(a, b, x) dx \right| \leq \frac{4|b - a|}{a} \int_0^\infty e^{-ax/2} dx = \frac{8|b - a|}{a^2}.$$

This will be less than  $\varepsilon$  provided  $|b - a|$  is smaller than the minimum of  $a^2\varepsilon/8$  and  $a/4$ .

(c) Since

$$\int e^{-ax} \sin x dx = -\frac{e^{-ax}}{a^2 + 1}(\cos x + a \sin x),$$

we have, using the result of part (b),

$$I'(a) = \int_0^\infty e^{-ax} \sin x dx = \frac{-1}{a^2 + 1}$$

for all  $a > 0$ .

(d) So now we know that  $I(a)$  satisfies  $I'(a) = -1/(a^2 + 1)$  and  $I(a) \rightarrow 0$  as  $a \rightarrow \infty$ . Together these imply

$$I(a) = \frac{\pi}{2} - \arctan a.$$

(e) Part (d) would seem to imply that  $I(0) = \pi/2$ . But all our reasoning in (a)-(d) is predicated on the assumption that  $a > 0$ . So we'll need a little argument to show that

$$\lim_{a \rightarrow 0^+} I(a) = \int_0^\infty \frac{\sin x}{x} dx$$

(we're exchanging the limit and the integral – again we'd be ok by some general theorem except that the interval of integration is infinite, and for  $a = 0$  the improper integral doesn't converge absolutely).

But the argument's pretty easy. Even though the integral of  $\sin x/x$  doesn't converge absolutely, it's easy to see that it converges, rather like the alternating series  $\sum(-1)^n/n$  (I'll leave the details of this to you).

So given  $\varepsilon$ , we can choose  $M$  large enough so that

$$\left| \int_M^\infty \frac{\sin x}{x} dx \right| < \frac{\varepsilon}{4}.$$

And since  $e^{-ax} \sin x/x$  has the same sign as  $\sin x/x$  for all  $x \geq 0$  but smaller absolute value, we have that

$$\left| \int_M^\infty e^{-ax} \frac{\sin x}{x} dx \right| < \frac{\varepsilon}{4}$$

as well.

Since  $e^{-Ma} \rightarrow 1$  as  $a \rightarrow 0$ , we can choose  $\delta$  sufficiently small that  $|1 - e^{-Ma}| < \varepsilon/(2M)$  provided  $|a| < \delta$ . This also implies  $|1 - e^{-ax}| < \varepsilon/(2M)$  for all  $0 \leq x \leq M$ . Therefore,

$$\left| \int_0^\infty e^{-ax} \frac{\sin x}{x} dx - \int_0^\infty \frac{\sin x}{x} dx \right|$$

is bounded above by

$$\int_0^M 1 - e^{-ax} dx + \left| \int_M^\infty e^{-ax} \frac{\sin x}{x} dx \right| + \left| \int_M^\infty \frac{\sin x}{x} dx \right| < M \frac{\varepsilon}{2M} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

provided  $0 < a < \delta$ . Therefore

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

7. Consider the equation

$$f(x, \varepsilon) = x^3 - 3x^2 + \varepsilon = 0.$$

- (a) Since  $f(x, 0) = x^3 - 3x^2 = x^2(x - 3)$ , the roots are 0 and 3.
- (b) Since we want  $x$  as a function of  $\varepsilon$ , we need  $\partial f/\partial x \neq 0$ . Since  $\partial f/\partial x = 3x^2 - 6x$  is zero for  $x = 0$ , the implicit function theorem doesn't apply. But  $\partial f/\partial x(3, 0) = 9$ , so the implicit function theorem gives a root  $x$  near 3 as a function of  $\varepsilon$  for  $\varepsilon$  in a neighborhood of 0.

(c) In general,

$$\frac{dx}{d\varepsilon} = -\frac{\partial f/\partial \varepsilon}{\partial f/\partial x}.$$

Since  $\partial f/\partial \varepsilon = 1$  for all  $x$  and  $\varepsilon$ , we get  $dx/d\varepsilon = -1/9$  for  $\varepsilon = 0$  and  $x = 3$ .

(d) The implicit function theorem only tells you that the solution  $x(\varepsilon)$  exists only for  $\varepsilon$  near zero. Now, let's show that the function  $x(\varepsilon)$  can be extended continuously (even smoothly) all the way to  $\varepsilon = 1$ . Do this as follows:

- (i) We have  $\partial f/\partial x = 3x^2 - 6x = 3x(x - 2)$ . Therefore, for any  $\varepsilon$  and for  $x > 2$ ,  $\partial f/\partial x \neq 0$ , and so the implicit function will apply to any pair  $(x, \varepsilon)$  with  $f(x, \varepsilon) = 0$  and  $x > 2$ , in particular for any  $(x, \varepsilon)$  for which  $f(x, \varepsilon) = 0$ ,  $0 \leq \varepsilon \leq 1$  and  $2.5 \leq x \leq 4$ . This gives the conclusion of part (d)(i).
- (ii) Since the implicit function theorem gives a solution  $x(\varepsilon)$  for  $\varepsilon$  in a neighborhood of the given value of  $\varepsilon$ , we have that the set of  $\varepsilon$  for which the solution exists is open. Since we are considering  $\varepsilon$  in the interval  $[0, 1]$ , we have that the set for which there is a solution is relatively open in  $[0, 1]$ .
- (iii) Since  $\partial f/\partial x > 0$  whenever  $x > 2$ , once we have fixed  $\varepsilon$ , then  $f$  is an increasing function of  $x$  for  $x > 2$  and so there can be at most one solution of  $f(x, \varepsilon) = 0$  with  $x > 2$  (by the mean-value theorem). Moreover, since  $f(2.5, \varepsilon) = -3.125 + \varepsilon$ , we have  $f(2.5, \varepsilon) < 0$  if  $0 < \varepsilon < 1$ . Also,  $f(4, \varepsilon) = 16 + \varepsilon > 0$  for  $0 < \varepsilon < 1$ , so we have at least one solution by the intermediate-value theorem. Hence, there is exactly one such solution  $x$  for each  $\varepsilon$ .
- (iv) Since  $\{x_i\}$  is a sequence of numbers in the compact interval  $[2.5, 4]$ , it must have a limit point. By Heine-Borel theorem, a subsequence converges to a point  $x^*$  in  $[2.5, 4]$ .
- (v) Since  $f(x_{\sigma_i}, \varepsilon_{\sigma_i}) = 0$  for all  $i$  (the  $\sigma_i$  is the index for the subsequence),  $x_{\sigma_i} \rightarrow x^*$ ,  $\varepsilon_{\sigma_i} \rightarrow \varepsilon^*$  and  $f$  is continuous, we must have  $f(x^*, \varepsilon^*) = 0$ .

- (vi) Parts (iv) and (v) show that the subset of  $\varepsilon \in [0, 1]$  for which there is a solution  $x(\varepsilon)$  contains all of its limit points, and therefore the subset is closed.
- (vii) Since the interval  $[0, 1]$  is connected, the only nonempty subsets both relatively open and closed in it are the empty set and  $[0, 1]$  itself. But our set of  $\varepsilon$  for which there is a solution is non-empty since it contains  $\varepsilon = 0$ . Thus we must have a continuous path of solutions  $x(\varepsilon)$  for  $0 \leq \varepsilon \leq 1$ .