**Ruminations on exterior algebra**

**1.1** *Bases, once and for all.* In preparation for moving to differential forms, let's settle once and for all on a notation for the basis of a vector space $V$ and its dual $V^*$. These are fairly standard notations.

For $V$, rather than having to say "...and suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $V$...", unless otherwise specified, we'll use either

$$\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots \frac{\partial}{\partial x^n}\right\}$$

or

$$\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$$

as our basis – the reason for the first of these will, I hope, become apparent next week. Of course, the first of these is a bit unwieldy, so we'll occasionally revert to the second. So the vector that might have been denoted $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ in $\mathbf{R}^3$ will now be

$$3\frac{\partial}{\partial x^1} + 4\frac{\partial}{\partial x^2} - 5\frac{\partial}{\partial x^3}$$

or $3\mathbf{e}_1 + 4\mathbf{e}_2 - 5\mathbf{e}_3$.

For the dual basis, we'll always use

$$\{dx^1, dx^2, \ldots, dx^n\}.$$

Sometimes, if we have two different coordinate systems or some kind of change of basis going on, we might also have $dy^1$, etc (and $\partial/\partial y^1, \ldots$ as well), but you get the picture. This isn't unwieldy, so we don't need an alternative notation.

In this notation we have, for example,

$$dx^1\left(3\frac{\partial}{\partial x^1} + 4\frac{\partial}{\partial x^2} - 5\frac{\partial}{\partial x^3}\right) = 3$$

and

$$\begin{aligned}(2dx^1 - 4dx^2 + dx^4)&(\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}^3)\\ &= 2dx^1(\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}^3) - 4dx^2(\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}^3) + dx^4(\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}^3)\\ &= 2 + 8 + 0 = 10.\end{aligned}$$

**1.2** *One-forms, two-forms, red-forms, blue-forms...* To construct the exterior algebra $\Lambda^*V$ means to construct a sequence of vector spaces $\Lambda^0V = \mathbf{R}$, $\Lambda^1V = V^*$, $\Lambda^2V, \ldots \Lambda^nV$. There are many ways to do this, but perhaps the simplest is to proceed as we did in class and begin by defining $\Lambda^p V$ as the set (vector space) of alternating $p$-linear functions on $V$. While that might sound a little unnerving, it just means that an element $\varphi$ of $\Lambda^p V$ is a map that takes $p$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ and returns a real number $\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_p)$. The "$p$-linear" part means that $\varphi$ is linear in each variable separately, in other words if all the vectors but $\mathbf{v}_i$ are kept fixed, and $\mathbf{v}_i$ is replaced by $a_1\mathbf{w}_1 + a_2\mathbf{w}_2$ (where $a_1$ and $a_2$ are numbers and $\mathbf{w}_1$ and $\mathbf{w}_2$ are vectors in $V$), then

$$\begin{aligned}\varphi(\mathbf{v}_1, \ldots, &\mathbf{v}_{i-1}, a_1\mathbf{w}_1 + a_2\mathbf{w}_2, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p) \\ &= a_1\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{w}_1, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p) + a_2\varphi(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{w}_2, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p).\end{aligned}$$

"Alternating" means that the order in which the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ appear inside $\varphi$ matters, and matters in a particular way. In an alternating function, if two of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are interchanged, then the value of $\varphi$ becomes the negative of what it had been before the interchange (in contrast to a *symmetric* function for which the interchange would have no effect, or a more general *tensor* where there is no rule governing what happens).

Elements of $\Lambda^p V$ are called $p$-forms (on $V$). We'll need some notation so that we can write down a few examples. Let's start with 2-forms. If $\varphi$ and $\psi$ are 1-forms, then we make a 2-form $\varphi \wedge \psi$ (called the "wedge product" of $\varphi$ and $\psi$) by defining

$$\varphi \wedge \psi(\mathbf{v}, \mathbf{w}) = \varphi(\mathbf{v})\psi(\mathbf{w}) - \varphi(\mathbf{w})\psi(\mathbf{v}).$$

This is clearly bilinear and alternating. For instance, $dx^1 \wedge dx^2$ is the 2-form for which $dx^1 \wedge dx^2(\mathbf{e}_1, \mathbf{e}_2) = 1$ (so $dx^1 \wedge dx^2(\mathbf{e}_2, \mathbf{e}_1) = -1$) and $dx^1 \wedge dx^2(\mathbf{e}_i, \mathbf{e}_j) = 0$ unless $\{i, j\} = \{1, 2\}$. In the same manner we can determine $dx^k \wedge dx^\ell$ for any $k$ and $\ell$. Note that if $k = \ell$ we must have $dx^k \wedge dx^\ell = 0$ (the zero bilinear function) since (using $k$ for both $k$ and $\ell$, since they're equal)

$$dx^k \wedge dx^k(\mathbf{v}, \mathbf{w}) = dx^k(\mathbf{v})dx^k(\mathbf{w}) - dx^k(\mathbf{w})dx^k(\mathbf{v}) = 0.$$

More generally, $\varphi \wedge \varphi = 0$ for any 1-form $\varphi$.

A bilinear form $\varphi$ on $V$ is determined by its values on all pairs of basis elements of $V$, i.e., by the $n^2$ numbers $\varphi(\mathbf{e}_i, \mathbf{e}_j)$ as $i, j = 1, \ldots, n$. Bescuse of the skew-symmetry, a 2-form is determined by its values on the $\binom{n}{2}$ numbers $\varphi(\mathbf{e}_1, \mathbf{e}_j)$ for $i < j$, since we already know that $\varphi(\mathbf{e}_i, \mathbf{e}_i) = 0$ and since $\varphi(\mathbf{e}_i, \mathbf{e}_j) = -\varphi(\mathbf{e}_j, \mathbf{e}_i)$. An easy linear algebra exercise, then, is to show that this implies that the set

$$\{dx^i \wedge dx^j \mid i = 1, \ldots, n-1,\ j = i+1, \ldots, n\}$$

is a basis for $\Lambda^2 V$, so we can write any 2-form as

$$\omega = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} dx^i \wedge dx^j.$$

Equivalently, we could set $a_{ji} = -a_{ij}$ if $i > j$ (and $a_{ii} = 0$) and write

$$\omega = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} dx^i \wedge dx^j.$$

We can continue in this way with $p$-forms for $p > 2$.

A 3-form $\omega$ is an alternating tri-linear function, so it's determined by the values $\omega(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$ for $i < j < k$. This is because we can get any permutation of $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$ by a sequence of swaps of two of them at a time, for instance:

$$\varphi(\mathbf{e}_k, \mathbf{e}_i, \mathbf{e}_j) = -\varphi(\mathbf{e}_i, \mathbf{e}_k, \mathbf{e}_j) = \varphi(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k).$$

As is to be expected from this, for any permutation $\sigma$ of $ijk$, we'll get

$$\varphi(\mathbf{e}_{\sigma(i)}, \mathbf{e}_{\sigma(j)}, \mathbf{e}_{\sigma(k)}) = \pm\varphi(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$$

according to the sign of the permutation $\sigma$ (plus if $\sigma$ is an even permutation and minus of $\sigma$ is odd).

If we start with three one-forms $\alpha$, $\beta$ and $\gamma$, we can make the 3-form $\alpha \wedge \beta \wedge \gamma$ by setting

$$\begin{aligned}\alpha \wedge \beta \wedge \gamma(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \alpha(\mathbf{v}_1)\beta(\mathbf{v}_2)\gamma(\mathbf{v}_3) &+ \alpha(\mathbf{v}_2)\beta(\mathbf{v}_3)\gamma(\mathbf{v}_1)\\ &+ \alpha(\mathbf{v}_3)\beta(\mathbf{v}_1)\gamma(\mathbf{v}_2) - \alpha(\mathbf{v}_1)\beta(\mathbf{v}_3)\gamma(\mathbf{v}_2)\\ &- \alpha(\mathbf{v}_3)\beta(\mathbf{v}_2)\gamma(\mathbf{v}_1) - \alpha(\mathbf{v}_2)\beta(\mathbf{v}_1)\gamma(\mathbf{v}_3).\end{aligned}$$

As with 2-forms, we get that a 3-form $\omega$ is determined by the $\binom{n}{3}$ numbers $\omega(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$ for $i < j < k$, so we can write

$$\omega = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} a_{ijk} dx^i \wedge dx^j \wedge dx^k,$$

or, after suitably defining $a_{ijk}$ so that it is completely skew-symmetric,

$$\omega = \frac{1}{6}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ijk} dx^i \wedge dx^j \wedge dx^k.$$

**1.3** *Formalism.* We could continue with the point of view of the preceding section, and regard elements of $\Lambda^p V$ as alternating $p$-linear functions, and there are certainly situations where it is profitable to take this point of view. Another perspective that is also useful in many situations is simply to regard the construction of $\Lambda^p V$ as a kind of algebraic game:

To play this game, we simply declare $\Lambda^0 V = \mathbf{R}$, then $\Lambda^1 V$ is the vector space with basis $\{dx^1, \ldots, dx^n\}$, in other words, the set of expressions of the form

$$a_1 dx^1 + \cdots + a_n dx^n$$

for any choice of constants $a_1, \ldots, a_n$. In this version, we don't worry about what $dx^i$ stands for, it's just a symbol.

We then declare that we can multiply the $dx^i$'s together in such a way that (1) the multiplication is associative, (2) the multiplication is alternating, (3) real coefficients always factor out of multiplications in the usual way (that is to say, we are constructing an $\mathbf{R}$-algebra), and (4) there aren't any other rules.

The upshot of all this is that we can multiply forms together the same way we multiply polynomials together, except for one small details: we're not allowed to permute the $dx^i$'s in a product unless we remember to apply the appropriate plus or minus sign (i.e., whenever we swap two of them we get a minus). To remind ourselves that the multiplication is a little odd, instead of just writing $dx^i\, dx^j\, dx^k \cdots$, we'll use the wedge symbol to denote the product: $dx^i \wedge dx^j \wedge dx^k \wedge \cdots$.

The entire algebra that we construct in this way is called $\Lambda^* V$. The elements of $\Lambda^p$ are simply the (homogeneous) degree-$p$ elements of the algebra. Working in the formalism becomes natural after a while, and you start to see patterns that you can take advantage of. For example, $\alpha \wedge \alpha = 0$ for any 1-form $\alpha$, and more generally,

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p = 0$$

if the $\alpha_i$ are linearly dependent (to prove this recall that if the $\alpha$'s are linearly dependent, then there are constants $c_1, \ldots, c_p$, not all zero, such that

$$c_1 \alpha_1 + \cdots + c_p \alpha_p = 0.$$

If we suppose $c_i \neq 0$, then we can express

$$\alpha_i = \frac{1}{c_i}(c_1 \alpha_1 + \cdots + c_{i-1}\alpha_{i-1} + c_{i+1}\alpha_{i+1} + \cdots + c_n \alpha_n).$$

If we replace $\alpha_i$ with the expression on the right in the product $\alpha_1 \wedge \cdots \wedge \alpha_p$, we see that we can then decompose the result into a sum of terms, each of which has some $\alpha_j$ that appears twice, and hence is zero. So the whole thing is zero. And because of rule number (4) above (that there are no other rules), the converse is true: if $\alpha_1, \ldots, \alpha_p$ are linearly independent, then their wedge product is non-zero in $\Lambda^p V$.

Just as with ordinary polynomial algebra, it is sometimes useful to try and "factor" or decompose a $p$-form as a product of forms of lower degree, and in particular as a product of 1-forms. To decide this becomes more complicated as $p$ increases and as $n$ (the dimension of $V$) increases.

For example, if $n = 2$, every 2-form is expressible as $c\,dx^1 \wedge dx^2$, which is clearly decomposable (as $(c\,dx^1) \wedge dx^2$, or $dx^1 \wedge (c\,dx^2)$ or any number of other things). It is perhaps less obvious but true that every 2-form over a 3-dimensional vector space is decomposable. For example, let

$$\omega = 3dx^1 \wedge dx^2 - 3dx^3 \wedge dx^1 + 3dx^2 \wedge dx^3.$$

To find $\alpha$ and $\beta$ such that $\omega = \alpha \wedge \beta$, we begin by observing that if such $\alpha$ and $\beta$ exist, then $\omega \wedge \alpha$ and $\omega \wedge \beta$ must both be zero. So let's see: if $\alpha = a_1 dx^1 + a_2 dx^2 + a_3 dx^3$, then $\omega \wedge \alpha = (3a_1 - 3a_2 + 3a_3) dx^1 \wedge dx^2 \wedge dx^3$ (right?) so let's choose $a_1 = 1$, $a_2 = 1$ and $a_3 = 0$. Next, we take $\beta = b_1 dx^1 + b_2 dx^2 + b_3 dx^3$, and then we have

$$\alpha \wedge \beta = (dx^1 + dx^2) \wedge (b_1 dx^1 + b_2 dx^2 + b_3 dx^3) = (b_2 - b_1) dx^1 \wedge dx^2 - b_3 dx^3 \wedge dx^1 + b_3 dx^2 \wedge dx^3.$$

In order for this to equal $\omega$, we must choose $b_3 = -3$ and $b_2 - b_1 = 3$. So we'll choose $\beta = 3dx^2 - 3dx^3$, and you can check that $\alpha \wedge \beta = \omega$.