Math 509 **Dr. DeTurck**
Final Exam **Wednesday, May 6**

1. (a) If $\mathbf{v}\colon S^2 \to \mathbf{R}^3$ is a function such that $\mathbf{v}(\mathbf{x})$ is perpendicular to $\mathbf{x}$, then $\mathbf{v}(\mathbf{x})$ points in a direction tangent to the sphere at $\mathbf{x}$. More computationally, since $\mathbf{x} \in S^2$, then $\mathbf{x} = (x_1, x_2, x_3)$ satisfies $x_1^2 + x_2^2 + x_3^2 = 1$. But then $\alpha = 2(x_1dx_1 + x_2dx_2 + x_3dx_3) = 0$. So for a vector $\mathbf{v}$ to be tangent to the sphere at $\mathbf{x}$, we must have that $\alpha(\mathbf{v}) = 0$, i.e., $x_1v_1 + x_2v_2 + x_3v_3 = 0$, i.e., $\mathbf{x} \cdot \mathbf{v} = 0$.

   (b) If $\mathbf{u}(\mathbf{x})$ is a *unit* tangent vector field to S^2, then $\mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) = 1$ for all $\mathbf{x} \in S^2$ and $\mathbf{u}(\mathbf{x}) \cdot \mathbf{x} = 0$ for all $\mathbf{x}$ (and $\mathbf{x} \cdot \mathbf{x} = 1$ of course). Let

$$\varphi(\mathbf{x}, t) = (\cos \pi t)\mathbf{x} + (\sin \pi t)\mathbf{u}(\mathbf{x}).$$

   Then

   - $\varphi(\mathbf{x}, t)$ is a smooth function of $\mathbf{x}$ and t becuase $\mathbf{u}$ is (and sine and cosine are)
   - $\varphi(\mathbf{x}, t) \in S^2$ for all $\mathbf{x}$, t, since

$$\begin{aligned}\varphi(\mathbf{x}, t) \cdot \varphi(\mathbf{x}, t) &= [(\cos \pi t)\mathbf{x} + (\sin \pi t)\mathbf{u}(\mathbf{x})] \cdot [(\cos \pi t)\mathbf{x} + (\sin \pi t)\mathbf{u}(\mathbf{x})] \\ &= (\cos^2 \pi t)\mathbf{x} \cdot \mathbf{x} + 2(\cos \pi t)(\sin \pi t)\mathbf{x} \cdot \mathbf{u}(\mathbf{x}) + (\sin^2 \pi t)\mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \\ &= (\cos^2 \pi t)(1) + 2(\cos \pi t)(\sin \pi t)(0) + (\sin^2 \pi t)(1) \\ &= 1\end{aligned}$$

   - $\varphi(\mathbf{x}, 0) = \mathbf{x}$, since $\cos(0) = 1$ and $\sin(0) = 0$.
   - $\varphi(\mathbf{x}, 1) = -\mathbf{x}$, since $\cos(\pi) = -1$ and $\sin(\pi) = 0$.

   So φ is a homotopy between the identity map $\mathbf{x} \mapsto \mathbf{x}$ and the antipodal map $\mathbf{x} \mapsto -\mathbf{x}$.

   (c) The volume (area) form on S^2 is $\omega = x_1dx_2 \wedge dx_3 + x_2dx_3 \wedge dx_1 + x_3dx_1 \wedge dx_2$, and so $\int_{S^2} \omega = 4\pi$. If $A\colon S^2 \to S^2$ is the antipodal map $A(\mathbf{x}) = -\mathbf{x}$, then by part (b), A is homotopic to the identity map. By problem 1 on the take-home part, this would imply that

$$\int_{S^2} A^*\omega = \int_{S^2} \omega = 4\pi.$$

   But $A([x_1, x_2, x_3]) = [-x_1, -x_2, -x_3]$, so

$$A^*\omega = (-x_1)d(-x_2) \wedge d(-x_3) + (-x_2)d(-x_3) \wedge d(-x_1) + (-x_3)d(-x_1) \wedge d(-x_2) = -\omega,$$

   and so

$$\int_{S^1} A^*\omega = \int_{S^2} -\omega = -4\pi.$$

   This contradiction means that the assumption that there is a nowhere-vanishing vector field on S^2 is incorrect.

   (d) Exactly the same reasoning applies to S^{2n} — the homotopy in part(b) is exactly the same, and for part(c), we use the volume form on $S^{2n} \subset \mathbf{R}^{2n+1}$:

$$\omega = x_1dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{2n+1} - x_2dx_1 \wedge dx_3 \wedge \cdots \wedge dx_{2n+1} + \cdots + x_{2n+1}dx_1 \wedge \cdots \wedge dx_{2n}.$$

Again, the take-home part would give $\int A^*\omega = \int \omega = \text{vol}(S^{2n})$ but since each term in ω has an odd number of factors, $A^*\omega = -\omega$, leading to the same contradiction.

(e) It's easy to check that the vector field on

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

defined by

$$\mathbf{v}(\mathbf{x}) = (-x_2, x_1, -x_4, x_3)$$

satisfies $\mathbf{v} \cdot \mathbf{x} = 0$ and $\mathbf{v} \cdot \mathbf{v} = 1$ for all $\mathbf{x} \in S^3$ (it is called a "Hopf vector field".)

2. Let

$$S_n = H_n - \ln n$$

and

$$a_n = (H_{n+1} - \ln(n+1)) - (H_n - \ln n) = \frac{1}{n+1} - \ln \frac{n+1}{n},$$

so that $S_1 = 1$ and $S_{n+1} = S_n + a_n$ for $n \geq 1$. Note that $-a_n$ is the area below the graph of $y = 1/x$ and above the horizontal line $y = 1/(n+1)$ and between the vertical lines $x = n$ and $x = n+1$. Since $1/(n+1) \leq 1/x \leq 1/n$ for x between n and $n+1$, we have that this area, and hence $|a_n|$ is less than $1/n - 1/(n+1) = 1/(n^2+n)$. Therefore, the series $\sum |a_n|$ converges by the comparison test, and so the limit of S_n, which is the sum of the series $\sum a_n$, is finite.

You can do better than this and get an estimate for $\lim S_n$. First, since the graph of $y = 1/x$ is concave up, the area described above that is equal to $-a_n$ is actually less than half of $1/(n^2+n)$. So

$$\sum_{n=K+1}^{\infty} (-a_n) < \sum_{n=K+1}^{\infty} \frac{1}{2} \frac{1}{n^2+n} = \frac{1}{2} \frac{1}{K+1},$$

since the latter is a telescoping series. Therefore, we know that $\lim S_n$ is bounded as follows:

$$\sum_{n=1}^{K} a_n > \lim_{n\to\infty} S_n > -\frac{1}{2} \frac{1}{K+1} + \sum_{n=1}^{K} a_n.$$

So, for example, if we take $K = 5$, we get (using Maple to calculate the logarithms and add the fractions) that $0.67 > \lim S_n > 0.57$. If you take $K = 1000$, then you get that γ is between 0.5770 and 0.5777.

3. Let $f(x) = \left\{ \frac{1}{x} \right\} \frac{x}{1-x}$.

(a) First, observe that we can rewrite f as follows:

$$f(x) = \begin{cases} \left(\frac{1}{x} - n\right) \frac{x}{1-x} & \text{for } \frac{1}{n+1} < x \leq \frac{1}{n},\ n \geq 2 \\ 1 & \text{for } \frac{1}{2} < x \leq 1 \end{cases}$$

Or, we could write f as:

$$f(x) = \frac{1}{2}\chi_{(\frac{1}{2},1]}(x) + \sum_{n=2}^{\infty} \left(\frac{1}{x} - n\right) \frac{x}{1-x} \chi_{(\frac{1}{n+1},\frac{1}{n}]}(x),$$

where $\chi_A(x)$ is the characteristic function of the set A. Since the fractional part of any number is certainly less than or equal to 1, and since $0 \le x/(1-x) \le 1$ for $0 \le x \le 1/2$, we see that $0 \le f(x) \le 1$ for all $x \in [0,1]$. Since the constant function 1 is summable on $[0,1]$, we can therefore use the dominated covergence theorem to interchange the integral and summation as follows:

$$\begin{aligned}
\int_0^1 f(x)\,dx &= \int_0^1 \frac{1}{2}\chi_{(\frac{1}{2},1]}(x) + \sum_{n=2}^{\infty}\left(\frac{1}{x}-n\right)\frac{x}{1-x}\chi_{(\frac{1}{n+1},\frac{1}{n}]}(x)\,dx \\
&= 1/2 + \sum_{n=2}^{\infty}\int_0^1 \left(\frac{1}{x}-n\right)\frac{x}{1-x}\chi_{(\frac{1}{n+1},\frac{1}{n}]}(x)\,dx \\
&= 1/2 + \sum_{n=2}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\frac{1}{x}-n\right)\frac{x}{1-x}\,dx
\end{aligned}$$

(b) In the chain of equalities above, we already observed that the first term in the sum for the integral is $1/2$.

(c) Now we have to calculate:

$$\begin{aligned}
\int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\frac{1}{x}-n\right)\frac{x}{1-x}\,dx &= \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\frac{1}{1-x}-\frac{nx}{1-x}\right)dx \\
&= \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\frac{1-n}{1-x}+n\right)dx \\
&= -(1-n)\ln(1-x)\Big|_{\frac{1}{n+1}}^{\frac{1}{n}} + nx\Big|_{\frac{1}{n+1}}^{\frac{1}{n}} \\
&= (n-1)\ln\frac{1-\frac{1}{n}}{1-\frac{1}{n+1}} + 1 - \frac{n}{n+1} \\
&= (n-1)\ln\frac{(n-1)(n+1)}{n^2} + \frac{1}{n+1} \\
&= (n-1)[\ln(n-1) - 2\ln n + \ln(n+1)] + \frac{1}{n+1}.
\end{aligned}$$

So the Nth partial sum of the series that evaluates the integral is

$$\frac{1}{2} + \sum_{n=2}^{N}(n-1)[\ln(n-1) - 2\ln n + \ln(n+1)] + \frac{1}{n+1}.$$

In the sum, for any given k, the quantity $\ln k$ will appear three times, first in the $k-1$st term with coefficient $k-2$, then in the kth term with coefficient $2(k-1)$ and then in the $k+1$st term with coefficient k. Since these three coefficients add up to zero, the terms involving $\ln k$ for all k up to $\ln(N-1)$ cancel, leaving only two logarithm terms. Also, there are $1/k$ terms starting with $1/2$ and ending at $1/(N+1)$. Therefore, using the H_n notation of the preceding problem, the Nth partial sum is

$$\begin{aligned}
&H_{N+1} - 1 + (N-1)\ln(N+1) + [(N-2) - 2(N-1)]\ln N \\
&\qquad = H_{N+1} - 1 + (N-1)\ln(N+1) - N\ln N \\
&\qquad = H_{N+1} - 1 + (N-1)\ln\frac{N+1}{N} - \ln N \\
&\qquad = H_N - \ln N + \frac{1}{N+1} - 1 + \ln\left(1+\frac{1}{N}\right)^N - \ln\left(1-\frac{1}{N}\right).
\end{aligned}$$

And using the result of the last problem, the fact that $e = \lim(1 + 1/n)^n$ and that $\ln e = 1$, we get that the limit of the Nth partial sum, and hence the value of the integral, is

$$\gamma + 0 - 1 + 1 + 0 = \gamma.$$

4. Let

$$A = \{x \in \mathbf{R} - \mathbf{Q} \,|\, \text{there are infinitely many fractions } p/q \text{ with } |x - p/q| < 1/q^3\},$$

and for each $q \geq 1$. let

$$B_q = \bigcup_{p=0}^{q} \left[\frac{p}{q} - \frac{1}{q^3}, \frac{p}{q} + \frac{1}{q^3} \right].$$

Clearly, every element of $A \cap [0, 1]$ is contained in infinitely many of the B_q. Also, $\mu(B_q) = 2/q^2$ and $\sum(2/q^2)$ converges, so the Borel-Cantelli lemma tells us that the set of x contained in infinitely many of the B_q's is zero. Since $A \cap [0, 1]$ is contained in this latter set, the measure of $A \cap [0, 1]$ is also zero. Moreover, every element of $A \cap [n, n + 1]$ is obtained by adding n to an element of $A \cap [0, 1]$, and so

$$A = \bigcup_{n=-\infty}^{\infty} (A \cap [n, n + 1]),$$

being a union of a countable number of sets of measure zero, has measure zero as well.

There's nothing particularly special about the exponent 3 in $1/q^3$ in this problem, except that 3 is greater than 2. The same result is true for any $p > 2$ – the set of irrational numbers x such that there are infinitely many fractions p/q in lowest terms with $|x - p/q| < 1/q^p$ has measure zero.

This notwithstanding, there are numbers, called *Liouville numbers*, which have the property that there are infinitely many fractions p/q with $|x - p/q| < 1/q^p$ for *all* values of p. One such number is 0.10100100000010000. . . , where the number of zeroes between successive 1's is $n!$. On the other hand, it can be shown that if γ is an algebraic irrational of degree n (this means that γ is a root of a polynomial of degree n with integer coefficients), then γ is not a Liouville number and one may choose $p = n$ in the condition above. Liouville used this observation to exhibit the first known transcendental number around 1850.