Math 509 **Dr. DeTurck**
Assignment 1 **Due Thursday, January 22**

This should be mostly review.

1. Let $f(t)$ be a continuous function on $[0, \infty)$ that satisfies $\lim_{t\to\infty} f(t) = L$. Compute
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,dt.$$
Justify your assertion.

2. Suppose $g(x)$ is a C^1 function from $\mathbf{R}$ to $\mathbf{R}$, such that $0 < g'(x) \le M < 1$ for all x.
   (a) Show that there is a point $x^* \in \mathbf{R}$ such that $x^* = g(x^*)$.
   (b) Show that there is only one such point.
   (c) Define the sequence $\{x_n\}$ via $x_0 = 0$ and $x_{n+1} = g(x_n)$ for all $n = 0, 1, 2, \ldots$. Show that $\lim_{n\to\infty} x_n = x^*$.

3. Consider the three sequences of functions $\{f_n(x)\}$, $\{g_n(x)\}$, and $\{h_n(x)\}$ from $[0, 1]$ to $\mathbf{R}$ defined by
$$f_n(x) = (1 - x)x^n$$
$$g_n(x) = n(1 - x)x^n$$
$$h_n(x) = n^2(1 - x)x^n.$$
What is the (pointwise) limit of each sequence? Which (if any) of the sequences converge uniformly? For which (if any) is it true that the limit of the integral of the functions in the sequence (from 0 to 1) is equal to the integral of the limit?

4. Let $\{x_n\}$ be a Cauchy sequence of points in $\mathbf{R}$, or in $\mathbf{R}^n$, or in the space of continuous functions on $[a, b]$ with the uniform norm, or in any metric space. Suppose a subsequence of $\{x_n\}$ converges to a point x. Show that the whole sequence also converges to x.

5. A set D in the plane is *arcwise connected* if for any two points in D there is a polygonal curve (i.e., one made up of straight line segments) in D joining them. Show that any *open* connected set is arcwise connected.

6. Let f be any function that has a Taylor series representation at 0 with radius of convergence 1, and let

$$T_n(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

denote the nth-degree Taylor polynomial of f. Find (with proofs of convergence etc.) the sum

$$\sum_{n=1}^{\infty} x^n (f(x) - T_n(x))$$

for $|x| < 1$.