Math 509 **Dr. DeTurck**
Assignment 2 **Due Thursday, January 29**

1. In class we proved the following lemma:

   Lemma. For n and k natural numbers, suppose $a_n(k)$ is a complex number, and suppose for each n we have

   $$\lim_{k\to\infty} a_n(k) = A_n$$

   is finite. Furthermore, suppose for each n there is a positive number M_n such that

   $$|a_n(k)| \le M_n$$

   for all n and k, and that

   $$\sum_{n=1}^{\infty} M_n < \infty.$$

   Then

   $$\lim_{k\to\infty} \sum_{n=1}^{\infty} a_n(k) = \sum_{n=1}^{\infty} A_n.$$

   Show that this lemma implies Theorem 8.3 in the textbook.

2. We also proved Euler's product formula for the sine function:

   $$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

   Use this to prove the following formula of Wallis:

   $$\sqrt{\pi} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \prod_{k=1}^{n} \frac{2k}{2k-1}.$$

3. Using only properties derived from the differential equation $y'' + y = 0$, prove that

   $$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

   and

   $$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

   and, by defining $\tan x = \sin x / \cos x$,

   $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

4. Prove that

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}$$

(well, this is almost true – explain what the exceptions might be).

5. Use the preceding problem to show

$$\arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

and from this derive

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

Then, use the series for the arctangent function to obtain an estimate of π accurate to six decimal places.

6. Generalize Niven's proof of the irrationality of π as follows:

Lemma. Let f be a continuous function on the interval $[0, T]$ which is positive on $(0, T)$. Suppose there are antiderivatives $f_1(t)$, $f_2(t)$,... with $f_1'(t) = f(t)$ and with $f_{n+1}'(t) = f_n(t)$ for all $n \geq 1$, such that $f_n(0)$ and $f_n(T)$ are integers for all $n \geq 1$. Then T is irrational.

Here is a sequence of steps that will result in a proof:

(a) Let P be the set of all polynomials $p(t)$ with real coefficients such that $g(0)$, $g(T)$, $g'(0)$, $g'(T)$,...,$g^{(n)}(0)$, $g^{(n)}(T)$,... are all integers. Prove that, if $g \in P$, then

$$\int_0^T f(t)g(t)\,dt$$

is an integer where f is the function in the lemma (use repeated integration by parts – differentiate g and integrate f).

(b) If $g \in P$ and $h \in P$ then so are $g + h$ and gh.

(c) If p and q are integers, then $p - 2qt \in P$. Also (use induction),

$$g_n(t) = \frac{t^n(p-qt)^n}{n!} \in P$$

for $n = 0, 1, 2, \ldots$. (This should begin to be reminiscent of Niven's paper).

(d) Use part (a) to conclude that, if T is rational, say $T = p/q$, then

$$\int_0^T f(t)g_n(t)\,dt$$

is a positive integer for all values of n.

(e) If M is the maximum value of $g_1(t) = t(p - qt)$ on $[0, T]$ and L is the maximum value for $f(t)$ on $[0, T]$, prove the estimate

$$0 < \int_0^T f(t) g_n(t)\, dt \leq T \cdot L \cdot \frac{M^n}{n!}.$$

(f) Consider what happens as $n \to \infty$, reach a contradiction, and conclude that T must be irrational.

7. Prove that if $0 < T < \pi$ and if $\cos T$ and $\sin T$ are both rational, then T is irrational. (Apply the preceding problem to the function $q \cdot \sin t$ if $T = p/q$.)

8. Prove that if x is positive and rational and $x \neq 1$, then $\ln x$ is irrational. (Apply the above problem with $T = \ln x$ and $f(t) = q \cdot e^t$.) What is the contrapositive of this statement?

9. Let's get even more ambitious: Prove the following lemma:

   Lemma. Suppose $g(x)$ is a polynomial with integer coefficients, and set $h(x) = x^n g(x)/n!$. Then $h^{(k)}(0)$ is an integer for $k = 0, 1, 2, \ldots$. Moreover, with the possible exception of the case $k = n$, the integer $h^{(k)}(0)$ is divisible by $n + 1$ (and if $g(0) = 0$, then $h^{(n)}$ must be divisible by $n + 1$ as well).

10. Prove that e is a transcendental number. This means that e is not a root of any non-zero polynomial over the integers, i.e., that

$$a_n e^n + a_{n-1} e^{n-1} + \cdots + a_1 e + a_0 = 0$$

for integers $a_0, a_1, \ldots, a_n$ implies that $a_0 = a_1 = \cdots = a_n = 0$.

Use the following outline of a proof of this by contradiction:

(a) Assume e is not transcendental and that e satisfies the relation above with (not all zero) coefficients $a_0, \ldots a_n$ of degree n. Without loss of generality (why?), assume $a_0 \neq 0$. Define

$$f(x) = \frac{x^{p-1}(x-1)^p (x-2)^p (x-3)^p \cdots (x-n)^p}{(p-1)!}$$

and

$$F(x) = f(x) + f'(x) + f^{(2)}(x) + \cdots + f^{(np+p-1)}(x)$$

where p is a prime number greater than 2 to be named later. Show that

$$\frac{d}{dx}(e^{-x} F(x)) = -e^{-x} f(x)$$

and

$$a_k \int_0^k e^{-x} f(x)\,dx = a_k F(0) - a_k e^{-k} F(k).$$

(b) Show that

$$\sum_{k=0}^{n} a_k e^k \int_0^k e^{-x} f(x)\,dx = -\sum_{k=0}^{n} \sum_{i=0}^{np+p-1} a_k f^{(i)}(k).$$

(c) Apply the lemma in the preceding problem to $f(x)$, $f(x+1)$,…,$f(x+n)$ (where you should use $p-1$ in place of the n in the lemma) to show that $f^{(i)}(k)$ is an integer for all values of i and k in the above sum, and that all of these integers are divisible by p except possibly for the single case $k=0$ and $i=p-1$.

(d) Show that

$$f^{(p-1)}(0) = (-1)^p(-2)^p \cdots (-n)^p,$$

so that $f^{(p-1)}(0)$ will not be divisible by p if we choose $p > n$. Furthermore if we choose $p > |a_0|$, then the double sum in (b) consists of a sum of multiples of p except for the term $-a_0 f^{(p-1)}(0)$. Why does this imply that both sides of (b) are (the same) non-zero integer(s)? (Why did p have to be prime?)

(e) Prove the following estimate:

$$\left|\sum_{k=0}^{n} a_k e^k \int_0^k e^{-x} f(x)\,dx\right| \leq \left(\sum_{k=0}^{n} |a_k|\right) e^n \frac{(n^{n+2})^{p-1}}{(p-1)!}.$$

(f) Explain why (e) contradicts (d) for sufficiently large p, and complete the proof.