**Math 509** **Dr. DeTurck**

**Assignment 3** **Due Thursday, February 5**

1. For each of the following functions, determine whether the limit exists as $(x, y) \to (0, 0)$. For those that do not have a limit, explain why not. For those that do, prove it by exhibiting a $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ if $0 < \sqrt{x^2 + y^2} < \delta$.

   (a) $f(x, y) = 2x^2 - 6xy + 5y^2$

   (b) $f(x, y) = \dfrac{x - y}{x^2 + y^2}$

   (c) $f(x, y) = xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right)$

   (d) $f(x, y) = \dfrac{x^2 + y}{(x^2 + y^2)^{1/2}}$

   (e) $f(x, y) = \dfrac{x^4 + y^4}{x^2 + y^2}$

2. For each of the functions in problem 1 that has a limit at (0,0), determine (and prove!) whether or not the function (with its removable discontinuity filled in) is differentiable at (0,0).

3. For $0 < x < \pi/2$ and $y > 0$, define:

$$u(x, y) = \frac{y}{\tan x}, \qquad v(x, y) = \frac{y}{\sin x}.$$

   (a) Calculate the derivative of $[u, v]$ with respect to $[x, y]$.

   (b) Solve for $x$ and $y$ in terms of $u$ and $v$. This defines a map from $[u, v]$ to $[x, y]$.

   (c) Calculate the derivative of $[x, y]$ with respect to $[u, v]$.

   (d) What is the relationship between the matrices you calculated in parts (a) and (c)? Prove it.

4. Prove that the function

$$f(x, y) = \begin{cases} (x^2 + y^2)\sin\left(\dfrac{1}{\sqrt{x^2 + y^2}}\right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

   *is* differentiable at $(0, 0)$ even though its partial derivatives (which do exist) are not continuous there (you should explain why the partial derivatives are discontinuous, too).

5. For which directions does

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

have directional derivatives at $(0,0)$? Is it differentiable there?

6. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be a function that is *homogeneous of degree* 1, i.e., $f(tx,ty) = tf(x,y)$ for all $(x,y) \in \mathbf{R}^2$. Show that if we set $x = r\cos\theta$ and $y = r\sin\theta$ (polar coordinates), then $f(x,y) = r\varphi(\theta)$ for some function $\varphi$ that is periodic with period $2\pi$. Under what conditions on $\varphi$ will $f$ be differentiable at $(x,y) = (0,0)$?

7. Let $\Omega$ be the open subset of $\mathbf{R}^2$ defined by removing the origin and the positive $y$-axis from the plane. Then define the function $\varphi$ on $\Omega$ by $\varphi(x,y) = y^2$ if $x > 0$ and $y \geq 0$, and $\varphi(x,y) = 0$ for all $(x,y)$ with either $x < 0$ or $y < 0$.

   (a) Show that $\varphi$ is differentiable at all points of $\Omega$.

   (b) Show that $\dfrac{\partial \varphi}{\partial x} = 0$ for all $(x,y) \in \Omega$, but that $\varphi$ is *not* independent of $x$.

   (c) Suppose that $\Omega \subset \mathbf{R}^2$ is an open set having the property that for each $y \in \mathbf{R}$ the set $\{x \in \mathbf{R} \,:\, (x,y) \in \Omega\}$ is either empty or is an *interval.* Prove that if $\varphi$ is a function on $\Omega$ such that $\dfrac{\partial \varphi}{\partial x} = 0$ everywhere on $\Omega$, then $\varphi$ is independent of $x$.

8. Assume $b > -1$. Consider the integral

$$\int_0^2 \frac{x^a - x^b}{\log x}\,dx$$

as a function of $a$. Notice that the value of the integral is zero when $a = b$. Take the derivative of the integral with respect to $a$ (justify the steps!), and use the result to evaluate the integral for any value of $a > b$.

9. Hey, some problems from the textbook! Page 239, problems 9 (you might want to refer back to a homework problem from the first set), 12, 15, 17